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A Basic Fuzzy-Estimation Theory for Available Operation of Complicated Large-Scale Network Systems

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Abstract—In this paper, we shall describe about a basic fuzzy-estimation theory based on the concept of set-valued operators, suitable for available operation of complicated large-scale network systems. Fundamental conditions for availability of system behaviors of such network systems are clarified in a form of β-level fixed point theorem for system of fuzzy-set-valued operators. Here, the proof of this theorem is accomplished by the concept of Hausdorff's ball measure of non-compactness introduced into the Banach space.

1. Introduction

In extremely complicated large-scale network systems, precise evaluation and perfect control, and also ideal operation, of overall system behaviors cannot be necessarily expected by using any type of commonplace technologies for maintenance, which might be accomplished by simple measure in usual hierarchical network structures.

In order to effectively evaluate, control and maintain those complicated large-scale networks, as a whole, the author has recommended to introduce some connected-block structure: *i.e.*, whole networks might be separated into several blocks which are carefully self-evaluated, self-controlled and self-maintained by themselves, and so, which are originally self-sustained systems. However, by always carefully watching each other, whenever they observe and detect that some other block is in ill-condition by some accidents, every block can repair and sustain that ill-conditioned block, through inter-block connections, at once. This style of maintenance of the system is sometimes called as locally autonomous, but the author recommends that only the ultimate responsibility on observation and regulation of whole system might be left for headquarter itself, which is organized over all blocks just as United States Government [1].

Here, let us consider Banach spaces X_i $(i = 1, \cdots, n)$ and Y_j $(j = 1, \cdots, n)$, and their bounded convex closed subsets $X_i^{(0)}$ and $Y_j^{(0)}$, respectively, corresponding to each block, Block i and Block j of whole network system. Let us introduce operators $f_{ij} : X_i \to Y_j$ such that $f_{ij}(X_i^{(0)}) \subset Y_j^{(0)}$ and let f_{ij} be completely continuous on $X_i^{(0)}$.

For each block : Block $i(i = 1, \cdots, n)$, dynamics of system behaviors can be represented originally by simple equations:

$$x_i = a_i f_{ii}(x_i), \quad (i = 1, \cdots, n), \tag{1}$$

where a_i is a continuous operator: $Y_i^{(0)} \to X_i^{(0)}$. These equations have solutions x_i^* in every $X_i^{(0)}(i = 1, \cdots, n)$, according to the well-known Schauder's type of fixed point theorem. Of course, these solutions represent original values of system behaviors.

f_{ii} represents the original performance of the i-th block itself, f_{ij} represents the operation fed-back through all other blocks ($j \neq i$) into the original i-th block, and f_{ji} represents inter-block connections from all other blocks, in order to repair and sustain the i-th block performance.

However, the fluctuation imposed on the actual system is nondeterministic rather than deterministic. In this case, even the effect due to a single cause is multi-valued, and the behavior is more naturally represented by a set of points, rather than a single point.

Therefore, it is reasonable to consider some suitable subset of the range of system behavior, in place of single ideal point, as target which the behavior must reach under influence of system control. Now, we can name it as an "available range" of the system behavior. Thus, by the available range, we mean the range of behavior, in which every behavior effectively satisfies good conditions beforehand specified, as a set of ideal behaviors. From such a point of view, the theory for fluctuation imposed on the system should be developed concerning the set-valued operator.

The author has given a series of studies on set-valued operators in functional analysis aspects, and has vigorously applied it to analysis of uncertain fluctuations of network systems [2], [3], [4].

Recently, the author gave a general type of fixed point theorem for the system of set-valued operator equations, in order to treat with extremely complicated large-scale network systems [1], [5], [7], [8].

Namely, let us introduce n set-valued operators $G_i : X_i \times \Pi_j^n Y_j \times \Pi^n Y_i \to \mathcal{F}(X_i)$ (the family of all non-empty closed compact subsets of X_i) $(i = 1, \cdots, n)$, where $\Pi_j^n Y_j$ means the direct product of n Y_j's, for any $j \in \{1, \cdots, n\}$, and $\Pi^n Y_i$ means direct product of n Y_i's, for fixed i.

Under some natural conditions, the author presented important fixed point theorems on systems of set-valued operator equations:

$$x_i \in G_i(x_i; f_{i1}(x_i), \cdots, f_{in}(x_i); f_{1i}(x_1), \cdots, f_{ni}(x_n)), \quad (i = 1, \cdots, n). \tag{2}$$

Proofs of fixed point theorems in ref. [1], [5] were accomplished by natural assumptions, on the other hand, the proof of the same theorem in ref. [7] was accomplished by a refined precise deduction, in weak topology, and the proof in ref. [8] was accomplished by the ball measure concept of non-compactness.

For convenience sake, let us define a direct product space $\boldsymbol{Y}_i \triangleq \prod_j^n Y_j \times \prod^n Y_i$ and also let $\boldsymbol{Y}_i^{(0)}$ be a non-empty bounded closed convex subset of $\boldsymbol{Y}_i^{(0)}$. Here, let us consider a vector $v_i \triangleq (x_i, \cdots, x_i;\ x_1, \cdots, x_n) \in V_i$ and an operator $f_i(v_i) :\ V_i \to \boldsymbol{Y}_i$ by

$$f_i(v_i) \triangleq (f_{i1}(x_i), \cdots, f_{in}(x_i);\ f_{1i}(x_1), \cdots, f_{ni}(x_n)) \ . \tag{3}$$

Here, we know that $y_{ij} \triangleq f_{ij}(x_i) \in Y_j$, $y_{ji} \triangleq f_{ji}(x_j) \in Y_i$ and $\boldsymbol{y}_i \triangleq (y_{i1}, \cdots, y_{in};\ y_{1i}, \cdots, y_{ni}) \in \boldsymbol{Y}_i$. Therefore, we have a simple representation of the system of set-valued operators (2), as follows:

$$x_i \in G_i(x_i;\ f_i(v_i)),\ (i = 1, \cdots, n). \tag{4}$$

On the other hand, the author recently presented a fixed point theorem for a general system of fuzzy-set-valued operator equations [6], under natural assumptions and with the proof similar to ref. [1] and [5].

Besides, the same fixed point theorem was proved precisely in weak topology in ref.

Further, in this report, we will present a refined estimation theory of the fixed point theorem for such a general system of fuzzy-set-valued operator equations, with a more basic proof by the use of the ball measure of non-compactness.

2. Fuzzy Set and Fuzzy-Set-Valued Operator

First of all, let us consider a family of all fuzzy sets originally introduced by Zadeh [10], in a Banach space X with the norm $\|\ \|$, and let any fuzzy set A be characterized by a membership function $\mu_A(x) : X \to [0, 1]$. Now, we can consider an α-level set A_α of the fuzzy set A as $A_\alpha \triangleq \{\xi \in X \mid \mu_A(\xi) \geq \alpha\}$, for any constant $\alpha \in (0, 1]$. The fuzzy set A is called compact, if all α-level sets are compact for arbitrary $\alpha \in (0, 1]$.

A fuzzy-set-valued operator G from X into X is defined by $G : X \to \mathcal{F}(X)$, where $\mathcal{F}(X)$ is a family of all non-empty , bounded and closed fuzzy sets in X. If a point $x \in X$ is mapped to a fuzzy set $G(x)$, the membership function of $G(x)$ at the point $\xi \in X$ is represented by $\mu_{G(x)}(\xi)$.

For convenience, let us introduce a useful notation: for an arbitrarily specified constant $\beta \in (0, 1]$, a point x belongs to the β-level set A_β of the fuzzy set A: $x \in A_\beta \triangleq \{\xi \in X \mid \mu_A(\xi) \geq \beta\}$ is denoted by $x \in_\beta A$ [11].

Here, let us introduce a new concept of β-level fixed point: for the fuzzy set $G(x)$, if there exists a point x^* such that $x^* \in_\beta G(x^*)$, then x^* is called β-level fixed point of the fuzzy-set-valued operator G [11].

Now, let us remember that we have introduced a new metric into the space of fuzzy sets [11, 12].

Definition 1 *Let us consider a Banach space X. For any fixed constant $\beta \in (0, 1]$, the β-level metric ρ_β between a point $x \in X$ and a fuzzy set A is defined as follows:*

$$\rho_\beta(x, A) \triangleq \inf_{\beta \leq \alpha \leq 1} d_\alpha(x, A), \tag{5}$$

where

$$d_\alpha(x, A) \triangleq \begin{cases} \inf_{y \in A_\alpha} \|x - y\| & \text{if } \alpha \leq \alpha_A, \\ \inf_{y \in A_{\alpha_A}} \|x - y\| & \text{if } \alpha > \alpha_A. \end{cases} \tag{6}$$

Here, $\alpha_A \triangleq \sup_{x \in X} \mu_A(x)$. And also, for any fixed constant $\beta \in (0, 1]$, by means of the Hausdorff metric d_H, the β-level metric $\mathcal{H}_\beta$ between two fuzzy sets A and B is introduced as follows:

$$\mathcal{H}_\beta(A, B) \triangleq \sup_{\beta \leq \alpha \leq 1} D_\alpha(A, B), \tag{7}$$

where D_α is defined as

$$D_\alpha(A, B) \triangleq \begin{cases} d_H(A_\alpha, B_\alpha) \\ \quad \text{if } \alpha \leq \min\{\alpha_A, \alpha_B\}, \\ d_H(A_{\alpha_A}, B_\alpha) \\ \quad \text{if } \alpha_A < \alpha \leq \alpha_B, \\ d_H(A_\alpha, B_{\alpha_B}) \\ \quad \text{if } \alpha_A \geq \alpha > \alpha_B, \\ d_H(A_{\alpha_A}, B_{\alpha_B}) \\ \quad \text{if } \alpha > \max\{\alpha_A, \alpha_B\}. \end{cases} \tag{8}$$

Here, $\alpha_B \triangleq \sup_{x \in X} \mu_B(x)$ and the Hausdorff metric d_H between two sets S_1 and S_2 is defined by

$$d_H(S_1, S_2) \triangleq \max\{\sup\{d(x_1, S_2) | x_1 \in S_1\}, \sup\{d(x_2, S_1) | x_2 \in S_2\}\},$$

where $d(x, S) \triangleq \inf\{\|x - y\| \mid y \in S\}$ is the distance between a point x and a set S.

In order to give a new methodology for the discussion more sophisticated than the one by usual set-valued operators, the author presented mathematical theories based on the concept of β-level fixed point, by establishing fixed point theorems for β-level fuzzy-set-valued nonlinear operators which describe detailed characteristics of such fuzzy-set-valued nonlinear operator equations, for every level $\beta \in (0, 1]$ [11, 12].

3. System of Fuzzy-Set-Valued Operator Equations

Now, let us introduce a more fine estimation theory for available operation of large-scale system of set-valued operators (2) and (4), by introducing β-level fuzzy estimation.

Originally, these sets are crisp. However, in order to introduce more fine estimation into these resultant fluctuation sets, here we can reconsider anew these sets G_i as fuzzy sets. Then, let us replace the above described crisp sets $G_i(x_i; f_i(v_i))$ by fuzzy sets with same notations, accompanied with suitable membership functions $\mu_{G_i}(\xi_i), \xi_i \in X_i$, which should be properly introduced corresponding to conscious planning for the fine evaluation of resultant fluctuations themselves.

In order to realize a more precise analysis, let us introduce different values of β as β_i $(i = 1, \ldots, n)$, consciously selected corresponding to every block : Block i.

Now, for any fixed constant $\beta_i \in (0, 1]$ $(i = 1, \ldots, n)$, we can introduce a system of β_i-level fuzzy-set-valued nonlinear operator equations:

$$x_i \in_{\beta_i} G_i(x_i; f_i(v_i)),\ (i = 1, \cdots, n). \tag{9}$$

If there exists a set of β_i-level fixed points $\{x_i^*\}$ in $X_i^{(0)}$ $(i = 1, \cdots, n)$, which satisfy the system of β_i-level fuzzy-set-valued operator equations (9), each x_i^* can be considered as a β_i-level likelihood behavior of Block i, $(i = 1, \cdots, n)$. Here, this β_i-level likelihood behavior x_i^* can be found in a closed domain in which the membership function $\mu_{G_i(x_i^*; f_i(v_i^*))}(\xi_i)$ has value larger than or equal to β_i.

4. Fixed Point Theorem For System of β_i-level Fuzzy-Set-Valued Operators

Here, we will present a fixed point theorem for such a general system of β_i-level fuzzy-set-valued operator equations.

Now, let us introduce a series of assumptions:

Assumption 1 *Let the operator $f_{ij} : X_i^{(0)} \to f_{ij}(X_i^{(0)}) \subset Y_j$ be completely continuous (continuous and compact).*

Assumption 2 *Let the fuzzy-set-valued operator $G_i : X_i^{(0)} \times \boldsymbol{Y}_i \to \mathcal{F}(X_i)$ (a family of all non-empty compact subsets of X_i) satisfies the following Lipschitz condition with respect to the β_i-level metric $\mathcal{H}_{\beta_i}$: that is, there are two kinds of constants $0 < k_i < 1$ and $h_i > 0$ such that for any $x_i^{(1)}, x_i^{(2)} \in X_i$, for any $\boldsymbol{y}_i^{(1)}, \boldsymbol{y}_i^{(2)} \in \boldsymbol{Y}_i$, G_i satisfies the inequality:*

$$\begin{aligned}&\mathcal{H}_{\beta_i}\left(G_i(x_i^{(1)}; \boldsymbol{y}_i^{(1)}), G_i(x_i^{(2)}; \boldsymbol{y}_i^{(2)})\right)\\ &\quad \le k_i \cdot \left\|x_i^{(1)} - x_i^{(2)}\right\| + h_i \cdot \left\|\boldsymbol{y}_i^{(1)} - \boldsymbol{y}_i^{(2)}\right\|.\end{aligned} \tag{10}$$

Now, we know that for any $x_i \in X_i^{(0)}$ and $f_i(v_i) \in \boldsymbol{Y}_i$, $G_i^{(0)}(x_i\ f_i(v_i)) \triangleq G_i(x_i;\ f_i(v_i)) \cap X_i^{(0)} \neq \phi$, and, moreover, there exist projection points $\tilde{x}_i{}' \in X_i^{(0)}$ of arbitrary point $x_i' \in X_i^{(0)}$ upon the set $G_{i\beta_i}^{(0)}(x_i\ f_i(v_i))$ such that

$$\left\|\tilde{x}_i{}' - x_i'\right\| = \min\left\{\left\|x_i' - z_i\right\| \,\middle|\, z_i \in G_{i\beta_i}^{(0)}(x_i\ f_i(v_i))\right\}, \tag{11}$$

where $G_{i\beta_i} \triangleq \{\xi \in X_i | \mu_{G_i}(\xi) \ge \beta_i\}$.

Then, we have the final result:

Theorem 1 *[Fixed Point Theorem] The system of β_i-level fuzzy-set-valued operator equations*

$$x_i \in_{\beta_i} G_i^{(0)}(x_i;\ f_i(v_i)), \quad (i = 1, \cdots, n) \tag{12}$$

has at least one fixed point $x_i^* \in X_i^{(0)}$.

5. The Proof of the Fixed Point Theorem

In order to prove the fixed point theorem; Theorem 1, on a basic aspect of mathematical foundation, let us introduce Hausdorff's ball measure of non-compactness χ, as follows: for the non-compactness of bounded subset S of real Banach space, Hausdorff's ball measure $\chi(S)$ is defined by[13]

$$\begin{aligned}\chi(S) \triangleq \inf\{&\epsilon \ge 0 \mid \\ &S \text{ can be covered with a finite number of} \\ &\text{balls of radii smaller than } \epsilon\}.\end{aligned} \tag{13}$$

Here, $\chi(S) = 0$ means that the closure of S is compact. Now, let us consider an arbitrary point $x_i \in X_i$, and the corresponding arbitrary point $\boldsymbol{y}_i \in \boldsymbol{Y}_i$. So, let us consider a bounded closed convex subset $X_i^{(0)} \subset X_i$, and the corresponding bounded closed convex subset $V_i^{(0)} \subset V_i$. Then, we have a lemma:

Lemma 1 *For an arbitrary point $x_i \in X_i^{(0)}$, we have*

$$\chi(G_i^{(0)}(x_i;\ f_i(V_i^{(0)}))) = 0, \tag{14}$$

where,

$$G_i^{(0)}(x_i;\ f_i(V_i^{(0)})) \triangleq \bigcup_{v_i \in V_i^{(0)}} G_i^{(0)}(x_i;\ f_i(v_i^{(0)})). \tag{15}$$

Eq. (14) means that the convex closure of $G_i^{(0)}(x_i;\ f_i(V_i^{(0)}))$ belongs to $\overline{\mathcal{F}(X_i)}$.

Next, let us define:

$$G_i^{(0)}(X_i^{(0)};\ f_i(V_i^{(0)})) \triangleq \bigcup_{x_i \in X_i^{(0)}, v_i \in V_i^{(0)}} G_i^{(0)}(x_i;\ f_i(v_i)). \tag{16}$$

Then, we have a lemma:

Lemma 2

$$\chi(G_i^{(0)}(X_i^{(0)};\ f_i(V_i^{(0)}))) \le k_i \cdot \chi(X_i^{(0)}). \tag{17}$$

Next, let us introduce some family of non-empty, convex, compact sets, invariant to the set-valued operator $G_i^{(0)}(x_i; f_i(v_i))$. For such a purpose, we shall prepare the following lemma:

Lemma 3 *Let $X^{(0)}$ be a bounded, closed and convex subset of the real Banach space X, and let the set-valued operator $F : X \to \mathcal{F}(X)$ be k-Lipschitz with respect to the non-compactness measure χ, with a Lipschitz constant $k : (0 < k < 1)$: i.e., for any convex subset $A \subset X^{(0)}$, there exists a constant $k(0 < k < 1)$ such that*

$$\chi(F(A)) \leq k \cdot \chi(A), \ (0 < k < 1). \tag{18}$$

If, let us introduce a sequence $\{W_m\}(m = 0, 1, 2, \cdots)$ by the successive procedure such that $W_0 = X_i^{(0)}$, and $W_m \triangleq conv.F(W_{m-1}), (m = 1, 2, \cdots)$, then we have in turn

$$X_i^{(0)} = W_0 \supset W_1 \supset W_2 \supset \cdots \tag{19}$$

and

$$\chi(W_m) \leq k^m \cdot \chi(W_0). \tag{20}$$

When $m \to \infty$, $\chi(W_m) \to 0$, and if we put $W_\infty \triangleq \bigcap_{m=0}^{\infty} W_m$, then, W_∞ is a non-empty, convex compact set, invariant to the set-valued operator F such that

$$F(W_\infty) \subset W_\infty. \tag{21}$$

From this flow of deduction, now, we can refer the well-known Ky Fan's fixed point theorem on upper semi-continuous set-valued operators, as follows:

Lemma 4 (Ky Fan [14]) *In a locally-convex topological linear space X, let V be its non-empty convex compact subset. Let a set-valued operator $H : V \to \mathcal{H}(V) \triangleq$ {Family of non-empty closed convex subsets} be upper semi-continuous. Then, there exists a fixed point x^* such that $x^* \in H(x^*) \subset V$.*

As a result, by Ky Fan's fixed point theorem, there exist fixed points $x_i \in_{\beta_i} G_i^{(0)}(x_i;\ f_i(v_0))$ in all subsets W_∞ with $V_{i\infty}^{(0)}(i = 1, \cdots, n)$.

6. Concluding Remarks

Thus, the fluctuation analysis of this type of large-scale network systems, undergone by undesirable uncertain fluctuations, can be successfully accomplished at arbitrary fine-level of estimation, by immediate application of the here-presented fixed point theorem for system of β_i-level fuzzy-set-valued nonlinear operators, with consciously selected different value of parameter β_i, for every Block *i*.

In this paper, the fixed point theorem was proved basically by using the concept of Hausdorff's ball measure of non-compactness.

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