

Spectral Analysis of Koopman Operator and Hamilton Jacobi Equation

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Extended Abstract

We establish a connection between the spectral theory of the Koopman operator and the solution of the Hamilton Jacobi (HJ) equation. The HJ equation occupies a central place in system theory, and its solution is of interest in various control problems, including optimal control, robust control, and input-output gain analysis. The HJ equation is a nonlinear partial differential equation, and its solution is at the heart of a data-driven reinforcement learning problem. The Koopman operator from the ergodic theory of dynamical system provides a linear representation of nonlinear system dynamics by lifting the system from the state space to the function space. The linear nature of the Koopman operator presents an opportunity to analyze the nonlinear dynamics through the spectrum, i.e., eigenvalues and eigenfunctions of this operator. The eigenfunctions and eigenvalues form the system's natural invariant and provide a diagonal and linear representation of the nonlinear dynamics in the Koopman eigenfunction coordinates. Apart from providing linear and diagonal representation of the nonlinear system, the Koopman eigenfunction can also be used in the characterization of the stable and unstable manifolds. The stable and unstable manifolds of the nonlinear system can be characterized in terms of the zero-level sets of the Koopman eigenfunctions with unstable and stable eigenvalues, respectively.

It is well known that one can associate a Hamiltonian system with the HJ equation. The solution of the HJ equation is intimately connected with the Hamiltonian system and can be obtained using the so-called Lagrangian submanifold of the Hamiltonian system. The main contribution of this work is in exploiting the spectral properties of the Koopman operator for the construction of Lagrangian submanifolds. We provide two different approaches for the construction of Lagrangian submanifolds. Our first approach uses Koopman eigenfunctions to decompose the Hamiltonian dynamical system associated with the HJ equation into an integrable and non-integrable form. As a result, the integrable part of the Hamiltonian system is resolved exactly, and the non-integrable part is approximately to construct the Lagrangian submanifold. The main highlight of this construction is that the Lagrangian submanifold is expressed as the function of the eigenfunction of the Koopman operator corresponding to the uncontrolled dynamical system. Hence, our first approach is ideal for the

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data-driven approximation of the HJ solution with application to data-driven optimal and robust control design for nonlinear systems.

In our second approach, we rely on the Koopman-based lifting of the Hamiltonian dynamical system to approximate the stable manifold and Lagrangian submanifold of the Hamiltonian system using the principal eigenfunctions of the Koopman operator. We present a convex optimization-based approach for the computation of the Koopman principal eigenfunctions. These eigenfunctions are then used in the approximation of the Lagrangian submanifold. We show that our proposed method for the approximate solution of the HJ equation in terms of the Koopman spectrum provides a natural extension of existing results from linear system theory to nonlinear systems. In particular, the Riccatti solution based on the linearization of the HJ equation can be obtained as a specific case of our proposed construction corresponding to the linear choice of basis function used in the lifting of the Koopman operator.

We compare the procedures for the approximate solution of the HJ equation based on the Koopman spectrum. We show that the second procedure based on Koopman lifting of the Hamiltonian dynamical system is more accurate than procedure one. However, the second procedure requires us to compute the eigenfunction of a larger, 2n, dimensional dynamical system compared to n dimensional dynamical system for procedure one, where n is the state space dimension of the dynamical system. Finally, we discuss the implication of the developed framework for solving optimal control, robust control, and reachability problem in the control dynamical system. In the following, we present simulation results for comparing optimal control obtained by solving the Riccatti equation based on linearized dynamics, i.e., linear quadratic regulator (LQR) control, procedure 1, procedure 2, and true optimal control.

Example

Consider the following example of 2D control oscillator system.

$$\dot{x}_1 = x_2 \dot{x}_2 = -x_1 + 0.105 x_2 + \frac{1}{2} x_2^2 x_1 + 1.1 x_1 x_2 + (1.1 + x_1) u$$
(1)

The objective is to design an optimal control with quadratic state cost and control cost i.e., $x_1^2 + x_2^2 + u^2$. The optimal control for this problem can be computed analytically and is of the form $u^* = -(1.1 + x_1)x_2$ and the optimal cost function is $V^* = x_1^2 + x_2^2$. The system was discretized using



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Euler method to obtain one-step time-series data with $1e^4$ initial conditions. Fig. 1. presents a comparison between linear part of the eigenfunctions and the approximate non-linear eigenfunctions. Similarly, in Fig. 2 we show the comparison of the feedback controller using LQR, Procedure 1, and Procedure 2. We notice that the optimal control obtained using procedure 2 shows a close match to the true optimal control. Comparison of the closed-loop trajectories for (-1.8, 1.8) initial condition is shown in the Fig. 3. In Fig. 4, we plot the control inputs and performance in terms of the cost. The optimal cost and the obtained using Procedure 2 shows a good match.



Figure 1: Nonlinear eigenfunctions, ϕ_1, ϕ_2 and the linear part of the nonlinear eigenfunctions w_1, w_2 .



Figure 2: Comparison of optimal feedback controllers obtained from LQR solution u_{LQR} , the true optimal control $u^{\star}(x)$, procedure 1 $u_{P_1}(x)$, and procedure 2 $u_{P_2}(x)$.



Figure 3: Comparison of state trajectories with LQR control input, Procedure 1, and Procedure 2.



Figure 4: Comparison of control inputs, and empirical cost function with LQR control input, procedure 1, and procedure 2.