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Existence and uniqueness of Koopman eigenfunctions near stable equilibria and limit cycles

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Abstract—The existence and uniqueness theory for smooth Koopman eigenfunctions, in the vicinity of an exponentially stable equilibrium or limit cycle, is described in the author's talk. This document is an attempt to briefly convey the flavor of some of these results and to illustrate, for equilibria, some of these in a simple setting.

1. Introduction

To the best of the author's knowledge, the state-of-the-art existence and uniqueness theory for C^k Koopman eigenfunctions ($1 \leq k \leq \infty$), in the vicinity of an exponentially stable equilibrium or limit cycle, was established in [1]. Simplified statements of these results and additional observations are contained in [2]. This document concentrates on the case of equilibria.

2. Principal Koopman eigenfunctions

A function is C^k if it is continuous and has continuous partial derivatives up to order k ($0 \leq k \leq \infty$). Consider an ordinary differential equation (ODE)

$$\dot{x} = f(x), \qquad x \in \mathbb{R}^n \tag{1}$$

where $f \colon \mathbb{R}^n \to \mathbb{R}^n$ is C^1.

Given an open set $U \subset \mathbb{R}^n$ that is forward invariant for f, a Koopman eigenfunction on U is a function $\psi \colon U \to \mathbb{C}$ satisfying, for some fixed $\lambda \in \mathbb{C}$,

$$\forall t \geq 0 \colon \psi(x(t)) = e^{\lambda t}\psi(x(0)) \tag{2}$$

along all trajectories $t \mapsto x(t)$ of f with initial condition $x(0) \in U$.

Assume now that U contains an equilibrium x_*, $f(x_*) = 0$, and that x_* is asymptotically stable with U contained in its basin of attraction. A Koopman eigenfunction $\psi \colon U \to \mathbb{C}$ is a *principal* eigenfunction if $\psi \in C^1$, $\psi(x_*) = 0$, and the derivative $d\psi(x_*) \neq 0$ at x_* is nonzero.

It is easy to show that, if ψ is a principal eigenfunction satisfying Eq. 2, then $d\psi(x_*)$ is a left eigenvector of the (complexified) Jacobian matrix $D_{x_*}f$ of f with eigenvalue λ. Thus, given an eigenvalue $\lambda \in \mathbb{C}$ of $D_{x_*}f$, natural questions arise: when do principal eigenfunctions satisfying Eq. 2 with λ exist? When are they uniquely determined by λ (up to scalar multiplication)? What is their level of smoothness?

Sufficient conditions for existence of C^1, $C^{1 \leq k \leq \infty}$, and C^ω (real-analytic) such principal eigenfunctions corresponding to *all* eigenvalues of $D_{x_*}f$ are readily derived from the standard Hartman, Sternberg, and Poincaré-Siegel linearization theorems, respectively. (The more well-known Grobman-Hartman C^0 linearization theorem does not produce C^1 eigenfunctions, and principal eigenfunctions are required to be C^1.) However, for *individual* eigenvalues λ of $D_{x_*}f$, existence of a corresponding $C^{1 \leq k \leq \infty}$ principal eigenfunction can be guaranteed under weaker conditions established in [1]. The level k of eigenfunction smoothness depends on properties of the eigenvalues of $D_{x_*}f$ and on the level of smoothness of f. The proof uses a contraction mapping, which also yields guarantees of convergence of iteration schemes (including Laplace averages) to principal eigenfunctions.

However, the question of conditions for *uniqueness* of principal eigenfunctions is arguably more interesting since, until recently, much less appeared to be known, except for C^ω eigenfunctions of C^ω vector fields [3]. Uniqueness conditions were recently obtained for $C^{1 \leq k \leq \infty}$ principal eigenfunctions in [1]. Like the existence conditions, these also involve properties of the eigenvalues of $D_{x_*}f$ and the level of smoothness of f. For an application of the uniqueness results of [1] to dynamical systems having certain sparsity structures, see [4, Thm 4.8, Cor. 4.9]

Finally, *all* C^∞ Koopman eigenfunctions—not only the principal ones—were classified for *generic* [2] C^∞ f in [1]: in this situation, every C^∞ Koopman eigenfunction on a forward invariant set U is a finite sum of finite products of principal eigenfunctions on U.*

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*In [1, 2] it was assumed for simplicity that the open set U is equal to the basin of attraction, but the results therein straightforwardly extend to any forward invariant open U contained in the basin. Also, the existence and uniqueness results described in the present document were actually obtained in [1] for general linearizing semiconjugacies—of which eigenfunctions are a special case—which are $C^{k,\alpha}_{\text{loc}}$, i.e. C^k with locally α-Hölder k-th partial derivatives, $0 \leq \alpha \leq 1$. One motivation for the refined $C^{k,\alpha}_{\text{loc}}$ smoothness considerations is that they make the principal eigenfunction existence and uniqueness results in [1] fairly close to optimal. Finally, the results of [1] also apply to discrete-time dynamical systems.

3. Uniqueness and existence examples

First consider the linear ODE

$$\dot{x} = -x, \qquad \dot{y} = -ky \tag{3}$$

on $\mathbb{R}^2$, where $k \geq 2$ is an integer. Here $x_* = (0,0)$ is a globally exponentially stable equilibrium point. Observe that both $\psi_1(x,y) := y$ and $\psi_2(x,y) := y + x^k$ are polynomial (hence C^ω, hence C^∞) principal eigenfunctions satisfying Eq. 2 with $\lambda = -k$, and ψ_1 and ψ_2 have the same derivative at x_*. Hence over any forward invariant open set U (e.g. $U = \mathbb{R}^2$), even polynomial principal eigenfunctions are *not* uniquely determined up to scalar multiplication by their derivatives at x_*.

This nonuniqueness is made possible since there is *resonance* between the system eigenvalues, which means that it is possible to write one eigenvalue as a linear combination of the others with nonnegative integer coefficients summing to at least 2 (cf. the displayed equation in [2, Prop. 11.(1)]). In this particular example, this is because $-k = k \cdot (-1)$ and $k \geq 2$ by assumption.

Next consider the linear ODE

$$\dot{x} = -x, \qquad \dot{y} = -ay \tag{4}$$

on $\mathbb{R}^2$, where $a > 1$ is not an integer. Here $x_* = (0,0)$ is again a globally exponentially stable equilibrium point. Because a is not an integer, resonance is no longer an issue.

However, observe that both $\psi_1(x,y) := y$ and $\psi_2(x,y) := y + |x|^a$ are $C^{\lfloor a \rfloor}$ principal eigenfunctions satisfying Eq. 2 with $\lambda = -a$, where $\lfloor \cdot \rfloor$ is the floor ("round down") function, and ψ_1 and ψ_2 have the same derivative at x_*. Hence over any forward invariant open set U (e.g. $U = \mathbb{R}^2$), $C^{\lfloor a \rfloor}$ principal eigenfunctions are *not* uniquely determined up to scalar multiplication by their derivatives at x_*. However, the results of [1] imply that such principal eigenfunctions *are* uniquely determined in this way if they are $C^{\lfloor a \rfloor + 1}$, i.e. one degree smoother. Moreover, the principal eigenfunction $\psi_2(x,y) := x$ satisfying Eq. 2 with $\lambda = -1$ is the unique such C^2 principal eigenfunction up to scalar multiplication.

These two uniqueness claims are guaranteed to hold because of nonresonance and because of the following asymmetric *spectral spread* property: when both -1 and $-a$ are multiplied by $(\lfloor a \rfloor + 1)$, numbers smaller than $-a$ are obtained; and when both -1 and $-a$ are multiplied by 2, numbers smaller than -1 are obtained, since $a > 1$ (cf. the first displayed inequality in [2, Prop. 11]).

Now suppose that $C^{\lfloor a \rfloor + 1}$ nonlinear (and also linear, if desired) terms are added to Eq. 4 without changing the eigenvalues of the system linearized at x_*. Then the results of [1] imply that, on any forward invariant open neighborhood U of x_*, there still exists a pair of $C^{\lfloor a \rfloor + 1}$ principal eigenfunctions ψ_1 and ψ_2 respectively satisfying Eq. 2 with $\lambda = -a$ and $\lambda = -1$. Moreover, ψ_1 is the unique $C^{\lfloor a \rfloor + 1}$ such function, and ψ_2 is the unique C^2 such function. Additionally, these eigenfunctions can be constructed via limiting procedures involving the flow of the nonlinearly perturbed ODE (see Eq. (26) and Rem. 14 of [1]).

All statements in this section can be justified using (Footnote $*$ and) the simplified version [2, Proposition 11] of the more general result [1, Proposition 6]. The examples above are adapted from [1, Example 2], which contains more details and nuances.

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