Cylindrical Korteweg-de Vries solitons on a ferrofluid jet

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Abstract—Dilute ferrofluids combine the hydrodynamic properties of Newtonian liquids with superparamagnetic response to external magnetic fields. For a cylindrical ferrofluid jet with a current-carrying wire along its axis the Rayleigh-Plateau instability may be suppressed due to the magnetic body force. The resulting axis-symmetric surface deformations show a linear dispersion relation similar to shallow water waves. Accordingly, the weakly non-linear regime is characterized by a Korteweg-de Vries (KdV) equation which can be derived using multiple scale perturbation theory. With the coefficients of this KdV-equation depending on the magnetic field strength both dark (depression) and bright (elevation) solitons are possible. These predictions have recently been verified in experiments, and also in a fully nonlinear numerical analysis.

1. Introduction

Solitons have fascinated experts and laymen alike ever since their spectacular discovery by Scott Russell in 1834 [1]. By an intricate balance between dispersion and non-linearity they move with constant speed and retain their shape even after mutual collisions. Although discovered in hydrodynamic systems they occur in a variety of settings reaching from non-linear optics, plasmas, crystal lattices, spiral galaxies to rigorous solutions in general relativity [2].

A particularly popular example for a soliton is provided by the solution of the Korteweg-de Vries equation [3]

$$\partial_t u(x,t) + 6u(x,t)\partial_x u(x,t) + \partial_x^3 u(x,t) = 0\,. \qquad (1)$$

Here $u(x,t)$ denotes the surface elevation of a liquid in a shallow duct as function of the space coordinate x along the duct and time t. The one-soliton solution of (1)

$$u(x,t) = \frac{c}{2}\,\mathrm{sech}^2\left(\frac{\sqrt{c}}{2}(x-ct)\right) \qquad (2)$$

describes a hump of constant shape moving to the right with velocity c. Eq. (1) results from a weakly non-linear analysis of surface waves if to linear order the system under consideration admits travelling wave solutions of the form $u \sim e^{i(kx-\omega t)}$ with dispersion relation

$$\omega = c_0 k + O(k^3) \qquad \text{for} \qquad k \to 0\,, \qquad (3)$$

where c_0 denotes the phase velocity. As is well known [4], surface waves on shallow water meet this requirement.

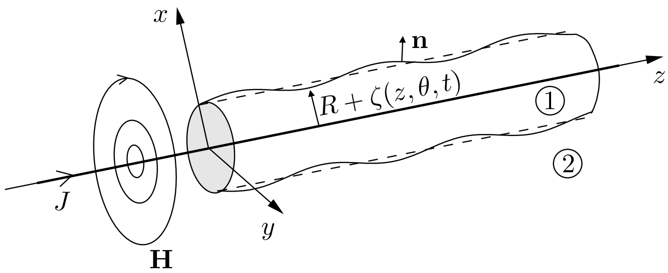

Figure 1: Schematic plot of the setup. A current-carrying wire is surrounded by a cylindrical ferrofluid with magnetic susceptibility χ and density ρ (region ①). Region ② is a nonmagnetic medium of negligible density treated as vacuum. The deflection of the surface from the cylindrical shape with radius R is denoted by $\zeta(z,\theta,t)$. The vector $\mathbf{n}$ denotes the normal on the free interface $R + \zeta(z,\theta,t)$.

In the present contribution we point out that the same holds true for axis-symmetric surface modulations of a ferrofluid jet under the influence of an azimuthal magnetic field resulting from a current flowing along the axis of the jet [5]. The general setup is shown in Fig. 1. Ferrofluids are stable suspensions of ferromagnetic nano-particles in Newtonian liquids and behave superparamagnetically in external magnetic fields [6]. A standard linear stability analysis of the cylindrical surface reveals that for sufficiently high current J the body force from the magnetic field suppresses the Rayleigh-Plateau instability of the jet [7, 8] and allows for travelling surface waves with dispersion (3). This in turn paves the way to derive a KdV equation for the weakly-nonlinear regime by multiple scale perturbation theory.

2. Linear stability

Let us list the basic assumptions of our theoretical analysis, for a detailed discussion see [5]. The current-carrying wire is thin, long and straight, the ferrofluid is incompressible, inviscid, and has density ρ, surface tension σ, and constant magnetic susceptibility χ. It is surrounded by a vacuum. We neglect gravity, and assume the flow to be irrotational.

In cylindrical coordinates (r,θ,z) the magnetic field is

$$\mathbf{H} = \frac{J}{2\pi r}\mathbf{e}_\theta\,. \qquad (4)$$

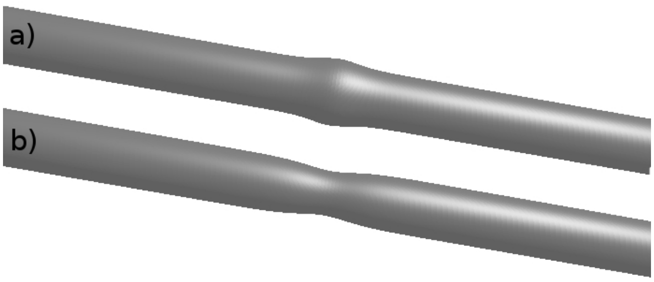

Figure 2: Schematic plot of a bright (a) and a dark (b) soliton as given by eq. (11).

The corresponding magnetic force, $\mathbf{F}_m = \mu_0(\mathbf{M}\nabla)\mathbf{H}$ attracts the ferrofluid radially inward. Here μ_0 denotes the susceptibility of the vacuum and $\mathbf{M} = \chi\mathbf{H}$ the magnetization of the fluid. Deviations from the cylindrical equilibrium shape of the ferrofluid with radius R are parametrized by the function $\zeta(z, \theta, t)$.

If we measure distances in units of R and times in units of $\sqrt{R^3\rho/\sigma}$ the magnetic field strength is characterized by the dimensionless Bond number

$$Bo = \frac{\mu_0\chi J^2}{4\pi^2\sigma R}\,. \tag{5}$$

We use the ferrohydrodynamic Bernoulli equation together with the magnetostatic Maxwell equations and their respective boundary conditions [6] to describe the coupling between the flow of the ferrofluid and the corresponding magnetic field configuration.

Linearizing these equations for small axis-symmetric surface deflections $\zeta(z, t) \ll 1$ of the form

$$\zeta(z, t) \sim \exp\Big(i(kz - \omega t)\Big) \tag{6}$$

one gets the dispersion relation [7, 8]

$$\omega^2(k) = k\frac{I_1(k)}{I_0(k)}\left(Bo - 1 + k^2\right), \tag{7}$$

where I_n denotes the Bessel function of index n with imaginary argument. In the long-wavelength limit, $k \to 0$, this gives rise to

$$\omega(k) = \sqrt{\frac{Bo - 1}{2}}\, k\left(1 - \frac{1}{16}\frac{Bo - 9}{Bo - 1}\, k^2\right) + O(k^5)\,. \tag{8}$$

Consequently, for $Bo > 1$ the Rayleigh-Plateau instability is suppressed and cylindrical surface waves with dispersion relation (3) may propagate along the jet.

3. Cylindrical solitons

On the basis of the results of the linear analysis one may now explore the interplay between the nonlinearity represented by higher order terms in ζ and the dispersion described by the $O(k^3)$ term in (8). This is conveniently done by multiple-scale perturbation theory; for details of the calculation see [5]. As a result one obtaines for the time evolution of the surface deflection the equation

$$\partial_t\zeta + c_0\partial_z\zeta + c_1\,\zeta\partial_z\zeta + c_2\,\partial_z^3\zeta = 0\,. \tag{9}$$

A similar equation was also obtained in [9]. The coefficients

$$c_0 = \sqrt{\frac{Bo - 1}{2}}\,, \quad c_1 = \frac{2Bo - 3}{4c_0}\,, \quad \text{and} \quad c_2 = \frac{Bo - 9}{32c_0} \tag{10}$$

all depend on the magnetic field strength Bo and hence, by changing the current in the wire, different regimes of the KdV equation may be investigated. The one-soliton solution of (9) is given by [9, 5]

$$\zeta(z, t) = \frac{3c}{c_1}\text{sech}^2\left(\sqrt{\frac{c}{4c_2}}(z - (c + c_0)t)\right). \tag{11}$$

where $c \ll 1$ is a free constant with the same sign as c_2. For $3/2 < Bo < 9$ we have $c < 0$ and $c_1 > 0$ and therefore (11) describes a dark or depression soliton with negative amplitude. For $1 < Bo < 3/2$ and $Bo > 9$ the amplitude is positive and (11) respresents a bright or elevation soliton which is much more common in hydrodynamic systems. Fig. 2 provides two respective examples.

4. Experimental verification

The experimental observation of the described solitons is non-trivial due to several complications. To realize zero gravity the ferrofluid column has to be surrounded by a non-magnetic liquid of the same density. However, then the hydrodynamics of this fluid has to be treated as well. Moreover, the above theoretical analysis assumes an inviscid fluid whereas real ferrofluids have appreciable viscosity resulting in a damping of all waves. For realistic parameter values the current J needs to be of the order of 100 A requiring a special cooling of the wire. Therefore, the assumption of a zero diameter for the current-carrying wire will be rather unrealistic.

Despite these (and other) problems cylindrical solitons of the discussed type have been observed recently in careful experiments [10]. First, the velocity and dispersion relation of linear waves were shown to be in agreement with the theoretical analysis. Then, by choosing the appropriate values of the current J depression as well as elevation solitons were observed. By a proper rescaling of the amplitude to account for the dissipative losses due to viscosity their shape was found to be well described by eq. (11), cf. Fig 3.

5. Fully nonlinear analysis

Very recently a thorough numerical analysis of the complete set of nonlinear equations describing the ferrohydrodynamics of the setup shown in Fig. 1 was performed [11].

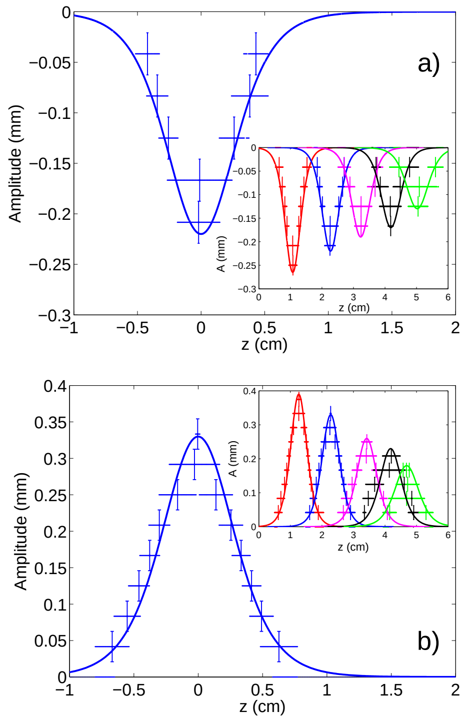

Figure 3: Experimental profiles of depression (top) and elevation (bottom) surface solitons shown by the crosses together with the scaled theoretical result given by eq. (11) shown as lines (from [10]).

The results of the weakly-nonlinear analysis discussed above were reproduced and extended into the region of strong nonlinearity. Moreover, in addition to the solutions accessible to the perturbative treatment of [5] new solutions were found that branch off discontinously from the uniform jet. Also, the situation with a non-zero radius of the current-carrying wire was treated. Given that in the experiments of [10] the radius of the wire was about 2/5 of the radius of the jet this is an important qualification of the results. Comparison between theory and experiment improves when the non-zero radius of the wire is taken into account. On the other hand, some experimental findings need a critical reinterpretation (for details see [11]).

6. Conclusion

Axissymmetric Korteweg-deVries solitons on the cylindrical surface of ferrofluid jets pose many interesting problems in non-linear ferrohydrodynamics, experimentally as well as theoretically and numerically. In the experiment the properties of the solitons can be tuned by changing the magnetic field, e.g., both elevation and depression waves may be studied with the same setup. On the theoretical side the effects of the outer fluid and the influence of viscous dissipation are challenging projects for future investigations.

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