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Chaos for Communication and Radar

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Abstract–Recent results that enable coherent communications and radar using chaotic waveforms are presented and discussed. Following the discoveries of chaos synchronization and control in the early 1990s, various engineering applications of chaos were envisioned. Novel communications technologies might use control to efficiently encode information in the symbolic dynamics of a chaotic oscillator, thereby exploiting the positive entropy for carrying information. Similarly, radar applications might exploit the positive entropy of a wideband, nonrepeating waveform for high-resolution imaging. As compelling as such applications are, subsequent development was hindered by the lack of a simple receiver that could compare favorably to standard correlation-based receivers. However, a new class of chaotic oscillators was recently reported that admit a linear representation and a simple matched filter. This development provides a coherent receiver to enable practical chaos communication and radar systems. Conceptual designs for both communications and radar using a matched filter for chaos are developed and considered.

1. Introduction

Following the discoveries of chaos synchronization and control in the early 1990s, various engineering applications of chaos were envisioned [1,2]. The large information capacity of chaotic waveforms suggested a potential use for high data rate communications [3]. Furthermore, the large bandwidth and aperiodicity of chaotic waveforms seemed to offer significant benefits for high range resolution radar [4]. However, practical realization of these applications was hindered by the lack of a simple coherent receiver that exploits the deterministic structure of these waveforms while operating in realistic environments.

In this paper, we reconsider these potential applications in light of the recent discovery of a class of exactly solvable chaotic oscillators [5]. These low-dimensional systems, which can be realized in a straightforward implementation, yield a provably chaotic waveform that is exactly represented by the linear convolution of a binary symbol sequence and a fixed basis function. Despite the chaotic nature of the waveform, the existence of a fixed basis function enables the development of a simple matched filter to provide an effective coherent receiver [6]. For chaos communications where the information is encoded in the symbolic dynamics via small perturbation control [3], a matched filter provides nearly optimal detection of the transmitted symbols. For chaos radar, a matched filter for a long waveform sequence offers a correlation receiver without the sampling and storage requirements of a comparable random signal waveform [7]. Consequently, this new class of exactly solvable chaotic oscillators may enable practical realization of chaos communications and radar technologies.

2. Solvable Chaos

We consider a hybrid system containing a continuous scalar state $u(t)$ and a discrete state $s(t)$ that was previously considered by Tsubone and Saito [8]. The continuous-time dynamics are described by the differential equation

$$\ddot{u} - 2\beta\dot{u} + \left(\omega^2 + \beta^2\right)\cdot\left(u - s\right) = 0 \tag{1}$$

where $\omega = 2\pi$ and $0 < \beta \le \ln 2$. Transitions in the discrete state are defined by the guard condition

$$\dot{u}(t) = 0 \quad \Rightarrow \quad s(t) = \operatorname{sgn}\left(u(t)\right) \tag{2}$$

meaning $s(t)$ is set to the sign of $u(t)$ whenever its time derivative vanishes, and $s(t)$ maintains this value until the guard condition is next met. Here, we define

$$\operatorname{sgn}(u) = \begin{cases} +1 & u \ge 0 \\ -1 & u < 0 \end{cases} \tag{3}$$

so that the switching state only takes the values $s(t) = \pm 1$.

Fig. 1 shows a typical solution obtained via numerical integration of the hybrid dynamical system for $\beta = \ln 2$. In this figure, the state $u(t)$ is the continuous waveform, while $s(t)$ is the square wave. Fig. 2 shows the corresponding phase-space projection. The solution waveform exhibits growing harmonic oscillations combined with random-like switching, and a dual-lobe structure is evident in the phase-space projection.

Recently, it was shown that an exact, analytic solution for this hybrid system can be found [5,6] The solution is the linear convolution

$$u(t) = \sum_{m=-\infty}^{\infty} s_m \cdot P(t - m), \quad s(t) = s_{[t]} \tag{4}$$

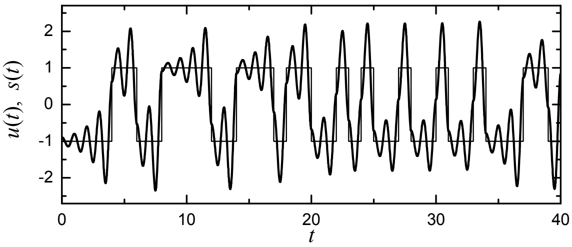

FIG. 1. Typical continuous waveform $u(t)$ and discrete state $s(t)$ from numerical integration of the hybrid dynamical system for $\beta = \ln 2$.

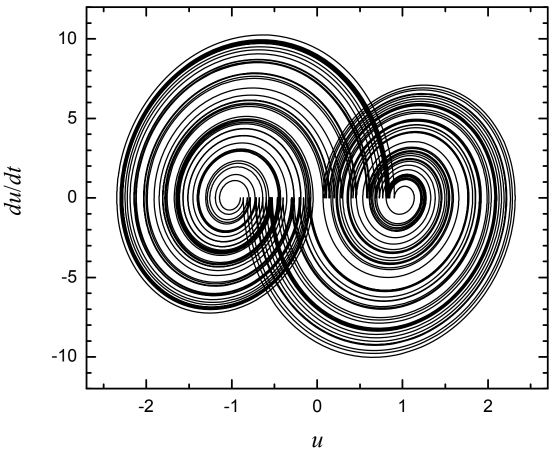

FIG. 2. Phase-space projection from numerical integration of the hybrid system for $\beta = \ln 2$.

where each $s_m = \pm 1$ and the square brackets indicate the greatest integer less than or equal to the argument. In the solution, each symbol s_m modulates a fixed basis function $P(t)$ centered at time $t = m$. Thus, it is right to think of the symbol s_m as the information emitted by the oscillator at time $t = m$, and that the oscillator emits one symbol with each unit of time. The fixed basis function is

$$P(t) = \begin{cases} \left(1 - e^{-\beta}\right) e^{\beta t} \left(\cos \omega t - \dfrac{\beta}{\omega} \sin \omega t\right), & t < 0 \\ 1 - e^{\beta(t-1)} \left(\cos \omega t - \dfrac{\beta}{\omega} \sin \omega t\right), & 0 \le t < 1 \\ 0, & 1 \le t \end{cases} \tag{5}$$

which is shown in Fig. 3 for $\beta = \ln 2$. We note that the solution (4) assumes a sequence of symbols that extends to $\pm\infty$; however, the form of the basis function (5) reveals that the state only depends on the current and future symbols ($m \ge [t]$).

As evident in the solution (4), an infinite sequence of symbols completely and uniquely specifies a solution to the hybrid system. Thus, the symbols also serve to label each possible solution. Returns in the continuous state at integer times $u_n = u(n)$ satisfy the recurrence relation

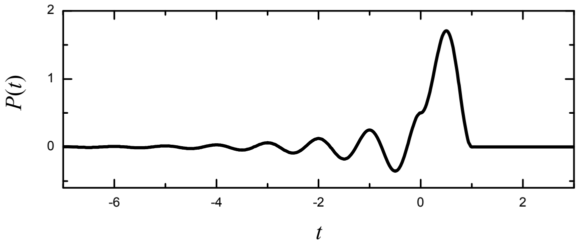

FIG. 3. Chaos basis function $P(t)$ for $\beta = \ln 2$.

$$u_{n+1} = e^{\beta} u_n - \left(e^{\beta} - 1\right) \cdot s_n \tag{6}$$

where

$$s_n = \operatorname{sgn}\left(u_n\right) \tag{7}$$

defines the concurrent symbol. Shown in Fig. 4, this return map is piecewise linear and characterized by the slope $e^{\beta} > 1$. As such, the iterated return map is chaotic, with Lyapunov exponent $\lambda = \beta$. Furthermore, the symbols s_n form a symbolic dynamics for the map, with a generating partition in equation (7).

The implication is that the hybrid dynamical system is also chaotic, and the symbols also form a symbolic dynamics for the hybrid oscillator. For $\beta < \ln 2$, the grammar is restricted and not every sequence of symbols is allowed; however, for the special case $\beta = \ln 2$, the map is a full shift and the grammar is unrestricted.

A matched filter for a given waveform is the optimal linear filter for detecting that waveform in AWGN. Practically, a matched filter can be realized as a filter with an impulse response that is the time reversal of the waveform to be detected. In particular, the matched filter for the basis function $P(t)$ is the linear operator $\mathcal{L}$ such that

$$\mathcal{L} \circ \delta(t) = P(-t) \tag{8}$$

where $\delta(t)$ is the Dirac delta function. This operator is realized by the stable filter

$$\dot{\eta} = v(t+1) - v(t) \tag{9}$$

$$\ddot{\xi} + 2\beta\dot{\xi} + \left(\omega^2 + \beta^2\right)\xi = \left(\omega^2 + \beta^2\right)\eta(t) \tag{10}$$

where $v(t)$ is the input, $\eta(t)$ is an intermediate state, and $\xi(t)$ is the output [6]. It is straightforward to verify that $v(t) = \delta(t)$ for $\xi(t) = P(-t)$. As such, equations (9)-(10) provide a matched filter for the basis function (5), and the matched filter can be used to detect the waveform symbols in the presence of noise.

3. Chaos Communications

The existence of a simple matched filter for a chaotic waveform enables development of practical chaos communications. Previously, it has been shown that an information sequence can be efficiently encoded in the symbolic dynamics of a chaotic waveform using chaos

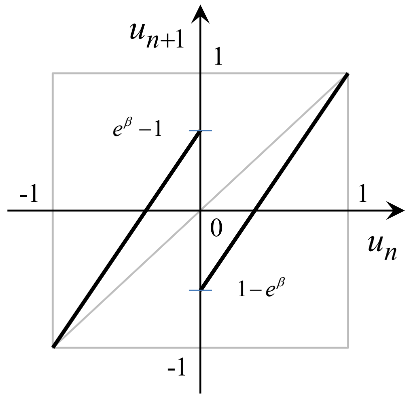

FIG. 4. Return map for analytic solution sampled at integer times.

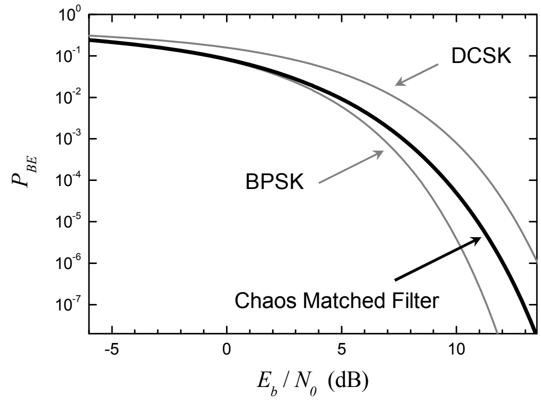

FIG 5. Analytic BER for matched filter chaos communications (β = ln 2), BPSK, and coherent DCSK.

control [3]. This elegant approach provides the advantage of encoding information in the natural dynamics of the chaotic oscillator; thus, the performance of a communication channel designed to transmit the chaotic waveform will not be degraded by the modulation. This feature is especially appealing for very high speed data communications where the data rate is commensurate with the carrier frequency, since the there are no modulation transients and the entire waveform can be utilized to carry information.

A matched filter for chaos provides a coherent receiver that achieves nearly optimal performance in the presence of channel noise. The theoretical performance characteristics of a chaos communications system incorporating the matched filter has been analyzed [6]. It is found that the bit-error rate (BER) for detecting symbols from an unconstrained oscillator in the presence of additive white Gaussian noise (AWGN) is

$$P_{BE} = \frac{1}{4(I_0-1)}\left\{(2I_0-1)\text{erfc}\left(\frac{2I_0-1}{\sigma\sqrt{2}}\right)-\text{erfc}\left(\frac{1}{\sigma\sqrt{2}}\right)\right\} - \frac{\sigma}{4(I_0-1)}\sqrt{\frac{2}{\pi}}\left\{\exp\left(-\frac{(2I_0-1)^2}{2\sigma^2}\right)-\exp\left(-\frac{1}{2\sigma^2}\right)\right\} \tag{11}$$

where

$$I_0 = 1+\left(1-e^{-\beta}\right)\frac{\omega^2-3\beta^2}{2\beta\left(\omega^2+\beta^2\right)} \tag{12}$$

is proportional to the energy contained in the basis function and σ^2 is the noise variance. Fig. 5 shows this result compared with binary phase-shift key (BPSK) encoding, where the horizontal axis

$$\frac{E_b}{N_0} = \frac{I_0^2}{2\sigma^2} \tag{13}$$

is the effective ratio of bit energy to noise power spectral density. For comparison, the best-case BER for coherent differential chaos shift keying (DCSK) is also shown in the figure [9].

4. Chaos Radar

The matched filter for a chaotic waveform can be extended in a standard way to provide coherent detection in a radar receiver [7]. To detect weak signals buried in noise, radar receivers rely on pulse compression, in which extended signals are detected using correlation. Larger bandwidth signals equate to higher range resolution, which is desirable. Random-signal radar is a relatively new technology that uses random noise-like waveforms to achieve very high range resolution.

Chaos naturally provides a large bandwidth signal that can also be used for high range resolution radar. Furthermore, the matched filter can be extended to detect a particular pulse sequence within the chaotic waveform. For this purpose, we consider any pulse sequence

$$Q(t) = \sum_{m=0}^{N-1} s_m \cdot P(t-m) \tag{14}$$

where the s_m are fixed. This waveform is analogous to a phase-coded, pulse compression waveform. The matched filter for this pulse sequence is then

$$\dot{\eta} = \sum_{m=0}^{N-1} s_m \cdot \left\{v(t+m+1)-v(t+m)\right\} \tag{15}$$

$$\ddot{\xi}+2\beta\dot{\xi}+\left(\omega^2+\beta^2\right)\xi = \left(\omega^2+\beta^2\right)\eta(t) \tag{16}$$

which simply requires additional evenly spaced delay taps weighted by the sequence symbols [7]. Importantly, no high fidelity copy of the transmitted signal is required for a correlation. Instead, the only information required from the transmitted waveform is the symbol sequence s_m.

In a random-signal radar, the emitted waveform must be digitized at a high rate (*i.e.*, the Nyquist requirement of twice the bandwidth) and resolution (*e.g.*, 8 bits). This high-fidelity representation is stored in a digital memory. Upon reception, the incoming signal is similarly digitized and a digital signal processor (DSP) performs the

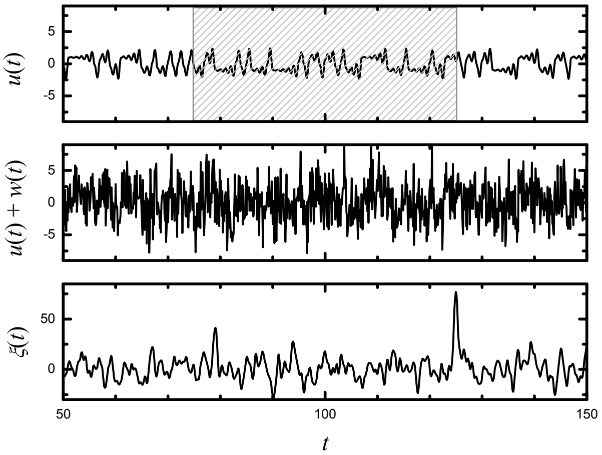

FIG. 6. Chaos radar simulation results.

correlation, usually using a fast Fourier transform (FFT). In contrast, the wideband waveform of a solvable chaotic oscillator can be compressed to a single bit per cycle (*i.e.*, the symbol sequence), which provides an order of magnitude reduction in the sampling and storage requirements. As such, the complexity and cost for a chaos radar should be less than that of a comparably performing random signal radar.

Fig. 6 shows the results of a chaos radar simulation using a matched filter receiver. The top plot shows a portion of a typical oscillator waveform that constitutes the transmitted waveform. A segment of 50 cycles is highlighted, from which symbols are extracted and used to weight the delay taps in a corresponding matched filter. To simulate a received waveform, the transmitted waveform is discretized at 10 samples per cycle, and random noise $w(t)$ is added to each sample from a Gaussian distribution with mean $\mu = 0$ and a standard deviation $\sigma = 2.5$. The middle plot shows the received signal, in which the chaos is visibly obscured by the noise. The received waveform is then used as an input to the matched filter. The output of the matched filter is shown in bottom plot. A prominent peak appears indicating that the transmitted waveform can be detected. We emphasize that this result was obtained by integration of the simple matched filter equations rather than an explicit evaluation of the correlation. Thus, this simulation indicates a simple analog receiver can provide effective pulse compression in chaos radar.

5. Conclusion

A simple matched filter for chaos is a significant component to enable development of practical chaos communications and radar technologies. For chaos communications, a matched filter provides a coherent receiver for detecting the information encoded in the symbolic dynamics of a transmitted waveform. For chaos radar, a matched filter enables significant pulse compression without the sampling and storage requirements of a comparable random signal waveform. Although other hurdles remain in the realization of practical chaos communications and radar, this development provides an important step toward making these technologies viable.

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