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Bifurcations in BVP Oscillators Coupled by Periodical Switching

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Abstract—Many systems with discrete events triggered by certain conditions, e.g., their states and/or time, have been proposed in various fields. We previously proposed a method to clarify bifurcation structures of the system including periodical switching event, because it is difficult to analytically solve bifurcation points/curves for such system. In this paper, by using our proposed method, we investigate bifurcation phenomena in BVP oscillators coupled by periodical switching device in both simulation and real circuit.

1. Introduction

Dynamical systems with intermittent activities generated by discrete events have been proposed or observed in various fields such as electric circuits, mechanical systems, and biological systems[1, 2]. From the standpoint of fundamental studies and engineering applications, it is very important to clarify a stability and a qualitative property of solutions in such systems. However, these systems include a non-differentiable point(s) in a solution orbit, and therefore, it is generally difficult to analytically solve the stability and bifurcation points of equilibria and periodic solutions.

In our previous study[3], we have proposed a shooting method to calculate local bifurcation points/curves for self-excited oscillators coupled by periodical switching device. As an application example of proposed method, we developed periodically switch-coupled BVP oscillators and showed the validity of the shooting method for circuit equations of this oscillators. However, detailed bifurcation structures of circuit equations are not clarified, and it is not investigated whether the real circuit shows same bifurcations. Therefore, we investigate detailed bifurcation structures of circuit equations and show that real circuit has same bifurcation structure.

2. Method

2.1. Periodically switch-coupled system

In this paper, we introduce a method to analyze local bifurcations of the system with discrete event consisting of two-dimensional autonomous systems coupled by periodical switching mechanism described as follows[3]:

$$\begin{cases} \dfrac{d\boldsymbol{x}_1}{dt} = \boldsymbol{f}_1(\boldsymbol{x}_1, \lambda) \\ \dfrac{d\boldsymbol{x}_2}{dt} = \boldsymbol{f}_2(\boldsymbol{x}_2, \lambda), \end{cases} \tag{1}$$

where $t \in \boldsymbol{R}$ and $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \boldsymbol{R}^2$ denote time and state vector of two oscillators, respectively. The solutions of each oscillator are represented as follows:

$$\begin{aligned} \boldsymbol{x}_1(t) &= (x_1(t), y_1(t)) = (\varphi_1(t, \boldsymbol{x}_{1k}, \lambda), \phi_1(t, \boldsymbol{x}_{1k}, \lambda)) \\ \boldsymbol{x}_2(t) &= (x_2(t), y_2(t)) = (\varphi_2(t, \boldsymbol{x}_{2k}, \lambda), \phi_2(t, \boldsymbol{x}_{2k}, \lambda)). \end{aligned} \tag{2}$$

Moreover, a solution of the whole system is represented as follows:

$$\boldsymbol{x}_k = (\boldsymbol{x}_{1k}, \boldsymbol{x}_{2k}), \tag{3}$$

where $\boldsymbol{x}_{1k}$ and $\boldsymbol{x}_{2k}$ are solutions at $t = kT$. In addition, at $t = kT$, x_{1k} and x_{2k} jump to a mean value of x_{1k} and x_{2k} by switching event, and then, y_{1k} and y_{2k} hold the value at $t = kT$. Hereinafter, we show a stability analysis of a periodic solution in this system.

2.2. Periodic solution and Poincaré map

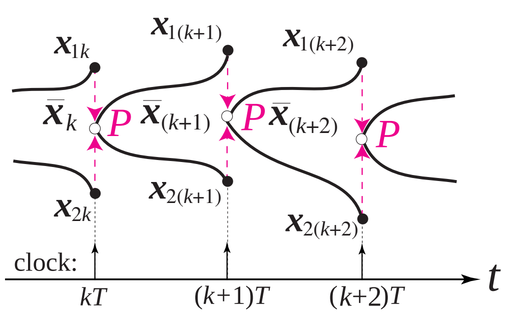

Figure 1: Schematic diagram of a solution in this model.

Let us consider a trajectory of the system in order to construct a Poincaré map. Figure1 shows a schematic illustrations of a trajectory of each oscillator. The maps of each system at $t = kT$, P_1 and P_2, are represented as follows:

$$\begin{aligned} &P_1 : \boldsymbol{R}^2 \rightarrow \boldsymbol{R}^2 \\ &\quad \boldsymbol{x}_{1k} \mapsto \overline{\boldsymbol{x}}_{1k} = \left(\overline{x}_{1k}, \overline{y}_{1k}\right) = \left(\frac{x_{1k} + x_{2k}}{2}, y_{1k}\right) \\ &P_2 : \boldsymbol{R}^2 \rightarrow \boldsymbol{R}^2 \\ &\quad \boldsymbol{x}_{2k} \mapsto \overline{\boldsymbol{x}}_{2k} = \left(\overline{x}_{2k}, \overline{x}_{2k}\right) = \left(\frac{x_{1k} + x_{2k}}{2}, y_{2k}\right), \end{aligned} \tag{4}$$

The map P of the whole system is defined as follows:

$$\begin{aligned} &P : \boldsymbol{R}^4 \rightarrow \boldsymbol{R}^4 \\ &\quad \boldsymbol{x}_k \mapsto \overline{\boldsymbol{x}}_k = (\overline{\boldsymbol{x}}_{1k}, \overline{\boldsymbol{x}}_{2k}) = \left((\overline{x}_{1k}, \overline{y}_{1k}), (\overline{x}_{2k}, \overline{y}_{2k})\right). \end{aligned} \tag{5}$$

The solution that left the map P behaves individually according to the dynamics of each system until $t = (k+1)T$. Therefore, The map of a solution orbit that left $\overline{\boldsymbol{x}}_k$ on the map P can be described as the following map S:

$$\begin{aligned} &S : \boldsymbol{R}^4 \rightarrow \boldsymbol{R}^4 \\ &\overline{\boldsymbol{x}_k} \mapsto \boldsymbol{x}_{(k+1)} = (\boldsymbol{x}_{1(k+1)}, \boldsymbol{x}_{2(k+1)}) \\ &\qquad = \left((x_{1(k+1)}, y_{1(k+1)}), (x_{2(k+1)}, y_{2(k+1)})\right), \end{aligned} \tag{6}$$

where the maps, S_1 and S_2, are written as follows:

$$\begin{aligned} &S_1 : \boldsymbol{R}^2 \rightarrow \boldsymbol{R}^2 \\ &\overline{\boldsymbol{x}}_{1k} \mapsto \boldsymbol{x}_{1(k+1)} = \boldsymbol{x}_1((k+1)T) = (x_{1(k+1)}, y_{1(k+1)}) \\ &\qquad = \Big(\varphi_{1(k+1)}(T, \overline{\boldsymbol{x}}_{1k}, \boldsymbol{\lambda}), \\ &\qquad\qquad \phi_{1(k+1)}(T, \overline{\boldsymbol{x}}_{1k}, \boldsymbol{\lambda})\Big) \\ &S_2 : \boldsymbol{R}^2 \rightarrow \boldsymbol{R}^2 \\ &\overline{\boldsymbol{x}}_{2k} \mapsto \boldsymbol{x}_{2(k+1)} = \boldsymbol{x}_2((k+1)T) = (x_{2(k+1)}, y_{2(k+1)}) \\ &\qquad = \Big(\varphi_{2(k+1)}(T, \overline{\boldsymbol{x}}_{2k}, \boldsymbol{\lambda}), \\ &\qquad\qquad \phi_{2(k+1)}(T, \overline{\boldsymbol{x}}_{2k}, \boldsymbol{\lambda})\Big). \end{aligned} \tag{7}$$

Eventually, the Poincaré map of the whole system is defined by a composite of two submaps (5) and (6) as follows:

$$\begin{aligned} &M : \boldsymbol{R}^4 \rightarrow \boldsymbol{R}^4 \\ &\boldsymbol{x}_k \mapsto \boldsymbol{x}_{(k+1)} = S \circ P(\boldsymbol{x}_k). \end{aligned} \tag{8}$$

A fixed point of the composite Poincaré map M satisfy the following equation:

$$M(\boldsymbol{x}_k) - \boldsymbol{x}_k = 0. \tag{9}$$

In general, this fixed point condition is not analytically solvable as usual for nonlinear systems. It can be numerically solved with respect to unknown variables $\boldsymbol{x}_k$ by using numerical methods such as Newton's method. Moreover, the following first-order derivatives of the map M are needed to calculate the fixed point by using Newton's method:

$$DM(\boldsymbol{x}_k) = \frac{\partial M}{\partial \boldsymbol{x}_k} = \frac{\partial S}{\partial \overline{\boldsymbol{x}}} \frac{\partial P}{\partial \boldsymbol{x}_k}$$

$$= \begin{bmatrix} \dfrac{\partial \varphi_{1(k+1)}}{2\partial \overline{x}_{1k}} & \dfrac{\partial \varphi_{1(k+1)}}{\partial \overline{y}_{1k}} & \dfrac{\partial \varphi_{1(k+1)}}{2\partial \overline{x}_{1k}} & 0 \\ \dfrac{\partial \phi_{1(k+1)}}{2\partial \overline{x}_{1k}} & \dfrac{\partial \phi_{1(k+1)}}{\partial \overline{y}_{1k}} & \dfrac{\partial \phi_{1(k+1)}}{2\partial \overline{x}_{1k}} & 0 \\ \dfrac{\partial \varphi_{2(k+1)}}{2\partial \overline{x}_{2k}} & 0 & \dfrac{\partial \varphi_{2(k+1)}}{2\partial \overline{x}_{2k}} & \dfrac{\partial \varphi_{2(k+1)}}{\partial \overline{y}_{2k}} \\ \dfrac{\partial \phi_{2(k+1)}}{2\partial \overline{x}_{2k}} & 0 & \dfrac{\partial \phi_{2(k+1)}}{2\partial \overline{x}_{2k}} & \dfrac{\partial \phi_{2(k+1)}}{\partial \overline{y}_{2k}} \end{bmatrix}. \tag{10}$$

The characteristic equation is described as follows:

$$\chi(\mu) = |\mu \boldsymbol{I}_4 - DM(\boldsymbol{x}_k)| = \boldsymbol{0}. \tag{11}$$

Note that eigenvalues μ of the Jacobian matrix correspond to the stability of the periodic solution. To calculate bifurcation parameters, the equations combined Eqs. (9) and (11) are simultaneously solved with respect to unknown variables $(\boldsymbol{x}, \lambda)$.

3. Periodically switch-coupled BVP oscillators

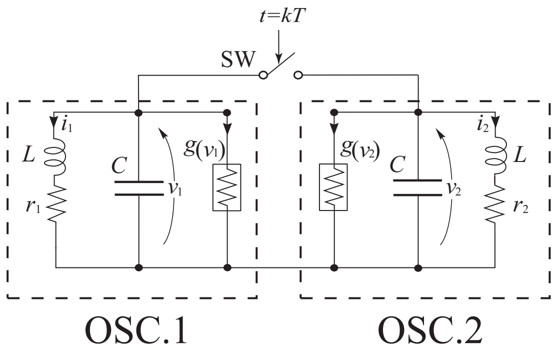

Figure 2: Periodically switch-coupled BVP oscillators.

Figure 2 shows the periodically switch-coupled BVP oscillators. The circuit equations of this system are described as the following equations:

$$\begin{cases} C\dfrac{dv_1}{dt} = -i_1 - g(v_1) \\ L\dfrac{di_1}{dt} = v_1 - r_1 i_1 \end{cases} \quad : \text{OSC.1} \tag{12}$$

$$\begin{cases} C\dfrac{dv_2}{dt} = -i_2 - g(v_2) \\ L\dfrac{di_2}{dt} = v_2 - r_2 i_2 \end{cases} \quad : \text{OSC.2.} \tag{13}$$

where v_1, v_2 and i_1, i_2 denote the voltages and currents of each BVP oscillators, respectively. The nonlinear resistors $g(v_i)$ are written by the following equation:

$$g(v_i) = -\alpha \tanh \beta v_i; \quad i = 1, 2. \tag{14}$$

By using the following variable transformation to Eqs.(12) and (13):

$$x_i = \frac{1}{\alpha}\sqrt{\frac{C}{L}}\, v_i,\ y_i = \frac{1}{\alpha}\, i_i,\ k_i = r_i\sqrt{\frac{C}{L}}; \quad i = 1, 2 \tag{15}$$

$$\tau = \frac{1}{\sqrt{LC}}t, \ \gamma = \alpha\beta\sqrt{\frac{L}{C}}, \tag{16}$$

we obtain the normalized equations:

$$\begin{cases} \dfrac{dx_1}{d\tau} = -y_1 + \tanh\gamma x_1 \\ \dfrac{dy_1}{d\tau} = x_1 - k_1 y_1 \end{cases} \tag{17}$$

$$\begin{cases} \dfrac{dx_2}{d\tau} = -y_2 + \tanh\gamma x_2 \\ \dfrac{dy_2}{d\tau} = x_2 - k_2 y_2. \end{cases} \tag{18}$$

The switching system is driven by the period-T external pulses. Solution orbits x_1 and x_2 of oscillators jump to the average value $(x_1 + x_2)/2$ when the switch is activated, because each oscillator has the capacitor of the same value. The variables y_1 and y_2 are not changed at the moment. On the other hands, When the switch is not activated, solution orbits behave individually by the dynamics of each oscillator. An example of solution orbits of this system is shown as Fig.3.

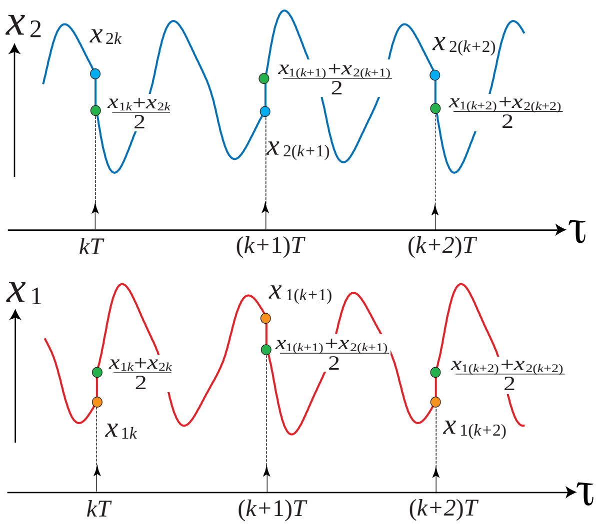

Figure 3: An example of solution orbits

By fixing the parameters $C = 22$[nF], $L = 10$[mH], $r_1 = 270[\Omega]$ and $r_2 = 458.5[\Omega]$, we obtain the parameter values $\gamma_1 = 1.567$, $\gamma_2 = 1.666$, $k_1 = 0.4$ and $k_2 = 0.68$, respectively. Moreover, we set the period of external pulse to $T = 200.0[\mu s]$. In this parameter, we shows a simulation result and behavior of real circuit in Fig.4.

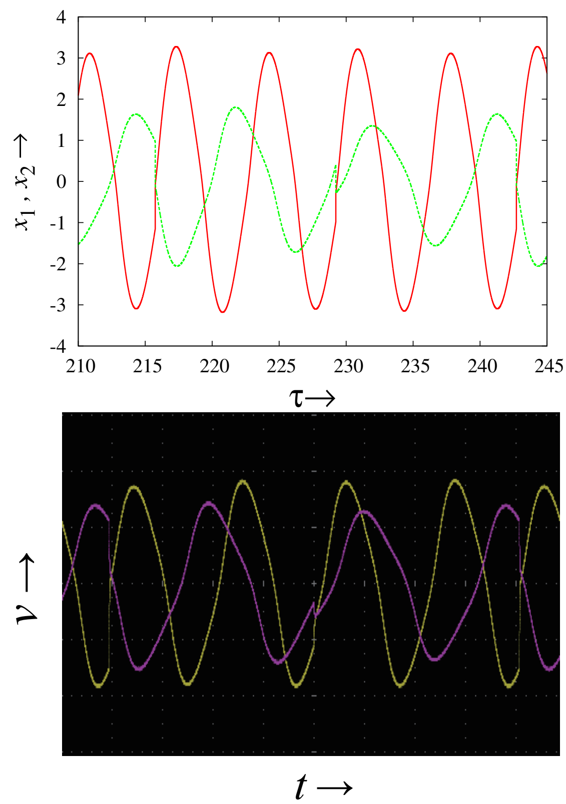

Figure 4: Simulation and experimental results at $r_1 = 270[\Omega](k_1 = 0.4)$ and $r_2 = 458.5[\Omega](k_2 = 0.68)$.

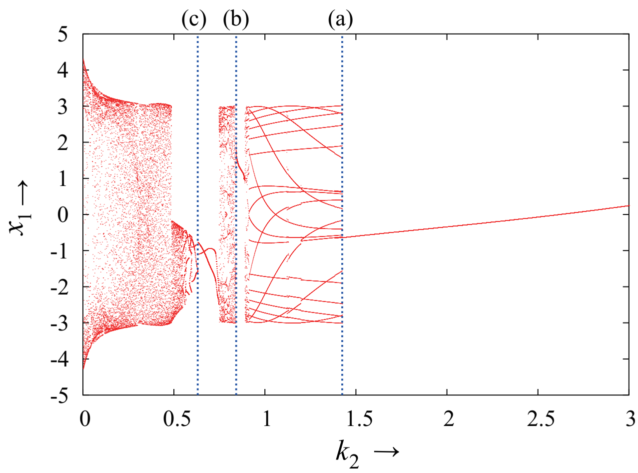

Figure 5: One-parameter bifurcation diagram at $k_1 = 0.4$.

4. Results

Figure 5 shows a one-parameter bifurcation diagram in x_1-k_2 plane when k_1 is fixed to 0.4. To confirm that the real circuit shows the same bifurcation structures, we investigate periodic solutions around three dashed lines (a), (b)

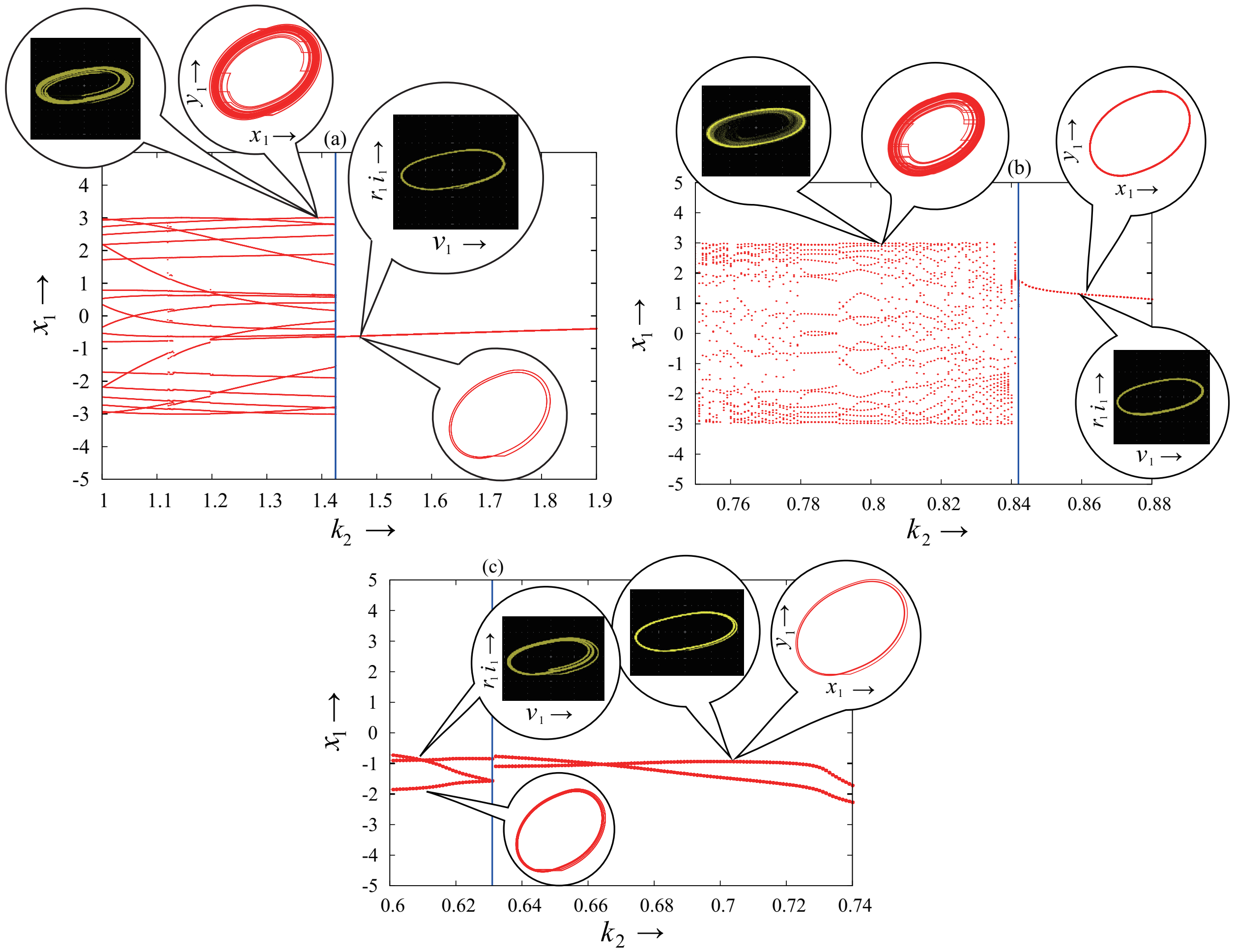

Figure 6: Bifurcations and some phase planes of the simulation and experiment.

and (c) as shown in Fig.6. By the saddle-node bifurcation of the periodic solution of the lines (a) and (b), we confirm that each period-1 solution is disappeared and then the system shows a chaos in both the simulation and experiment. Moreover, By the period-doubling bifurcation of the line (c), period-2 solution changes to a period-4 solution, but this solution is almost simultaneously disappeared by the saddle-node bifurcation and then the state of this system shows a period-3 solution. From these results, we clarify that the circuit equation and the real circuit have the same bifurcation structure and verify the validity of our proposed method for this system.

5. conclusion

In this paper, we solved a one-parameter bifurcation diagram and confirmed that the circuit equation and the real circuit have the same bifurcation structure. From the result, the validity of our proposed method in both simulation and experiment is verified. As a future works, we will solve some two-bifurcation diagrams to clarify more detailed bifurcation structures by using our method.

Acknowledgments

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