Generalized Liouville Equation for Nonlinear Dynamic Circuits

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Abstract—In this paper, the generalized Liouville equation by *Gerlich* is used to study the transient behaviour near limit cycles during an entire ensemble of solutions is considered.

1. Introduction

We consider the dynamics of nonlinear circuits which is defined by the solution set of the corresponding state space equations [2]. To obtain an unique solution of the state space equations, we have to prescribe specific initial values. In order to include this uncertainty, we use the ensemble approach of statistical mechanics [3]. For this purpose, we consider not only a single solution of the state space equations but an entire set of solutions that satisfy the same state space equations but with different initial values. Therefore, we use a density function for the initial values to characterise such an ensemble of solutions and which can be interpreted as a weighting procedure. The time dependence of the density function can be defined by using the state space equations of a circuit.

It is shown in statistical mechanics that for energy preserved systems the density function fulfils the Liouville equation. The main consequence is that the volume in state space is preserved. However, electrical circuits include in almost cases dissipative circuit elements such that this concept has to be generalized. Hence, in our paper we use the concept of a generalized Liouville equation by *Gerlich*[1].

2. State Space Equations and Ensemble Description

Generically, the n-dimensional equations of motion of dynamical systems can be described by ordinary differential equations of first-order, cf. *Chua and Lin* [2]

$$\dot{\mathbf{x}}(t) = \mathbf{f}\left(\mathbf{x}(t), t\right) \tag{1}$$

with $\mathbf{x}(t) \in \mathbb{R}^n$ as integral quantities, such as voltages $\mathbf{v}(t)$ or currents $\mathbf{i}(t)$, and $\mathbf{f}\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$. If $\mathbf{f}$ satisfies a Lipschitz condition the (local) dynamics of an electrical network can be described by all solutions

$$\mathbf{x}(t) = \mathbf{h}(\mathbf{x}_0, t) \tag{2}$$

where the initial values $\mathbf{x}_0$ are chosen from an admissible set $D \subseteq \mathbb{R}^n$. Alternatively, the initial value problem (1) with a prescribed initial value can be replaced by the entire set of solutions of (2)

$$\mathcal{L} := \{\mathbf{x}(t) = \mathbf{h}(\mathbf{x}_0, t) | \, \mathbf{x}_0 \in D \subseteq \mathbb{R}^n\}. \tag{3}$$

A mathematical characterization of $\mathcal{L}$ is difficult and even an analytical or numerical construction of the dynamics of a set of initial values $\tilde{D} \subset D$ cannot be realized in a suitable manner. Therefore, we will study the evolution of density functions $p(\mathbf{x}, t)$ defined on D in state space $\mathbb{R}^n$. For this purpose, a partial differential equation for $p_0\colon D \to \mathbb{R}$ is derived. This concept was developed by *Gerlich* [7] and the equation is called generalized Liouville equation in analogy to the Liouville equation in Hamiltonian mechanics; cf. *Goldstein et al.* [8]. Therefore, let p be the probability density for a state variable $\mathbf{x}$ at time t then the expected value or mean $\langle\cdot\rangle$ of an arbitrarily scalar function $\mathbf{g}$ is calculated according to

$$\langle \mathbf{g}(\mathbf{x}(t)) \rangle = \int \mathbf{g}(\mathbf{x})\, p(\mathbf{x}, t)\, \mathrm{d}\mathbf{x}. \tag{4}$$

The time derivative of $\mathbf{g}$ obeys with aid of the chain rule and Eq. (2)

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{g}(\mathbf{x}, t) = \frac{\partial \mathbf{g}}{\partial t} + \sum_{i=1}^{n} \frac{\partial x_i}{\partial t}\frac{\partial \mathbf{g}}{\partial x_i} = \frac{\partial \mathbf{g}}{\partial t} + \sum_{i=1}^{n} f_i \frac{\partial \mathbf{g}}{\partial x_i}. \tag{5}$$

Let $\mathbf{J}(\mathbf{x}_0, t) = \partial \mathbf{h}/\partial \mathbf{x}_0$ be the Jacobi matrix whose nonzero Jacobian determinant ensures the variable transformation $\mathbf{x} \leftrightarrow \mathbf{x}_0$ so that a time independent function $\tilde{p}(\mathbf{x}_0)$ with the abbreviation $\tilde{J} := \det^{-1}(\mathbf{J})$ obeys

$$\mathbf{g}(\mathbf{x})\, \tilde{p}(\mathbf{x}_0)\, \tilde{J}(\mathbf{x}_0)\Big|_{\mathbf{x}_0 = \mathbf{h}^{-1}(\mathbf{x}, t)} \mathrm{d}\mathbf{x} = \mathbf{g}(\mathbf{x})\, \tilde{p}(\mathbf{x}_0)\Big|_{\mathbf{x} = \mathbf{h}(\mathbf{x}_0, t)} \mathrm{d}\mathbf{x}_0. \tag{6}$$

Hence, it applies $p = \tilde{p}\,\tilde{J}$. The time derivative of $\tilde{p}$ in accordance with Eq. (5) and a subsequent multiplication with the inverse Jacobian determinant yields

$$0 = \tilde{J}\left(\frac{\partial}{\partial t} + \sum_{i=1}^{n} f_i \frac{\partial}{\partial x_i}\right)\tilde{p}. \tag{7}$$

If f is twice continuously differentiable, it follows after some calculations with $\partial h_i/\partial t = f_i(\mathbf{x}, t)|_{\mathbf{x}=\mathbf{h}(\mathbf{x}_0, t)}$ the generalized Liouville equation

$$\frac{\partial}{\partial t} p(\mathbf{x}, t) + \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(f_i(\mathbf{x}, t)\, p(\mathbf{x}, t)\right) = 0. \tag{8}$$

[1] Note, if the electrical circuit is in permanent contact with the stochastic heat bath, we have to use the Fokker-Planck equation [3].

The Liouville equation is a first-order quasilinear partial differential equation. Quasilinear means that the coefficients may depend on p, but not on any derivatives of p. This equation can be solved by the method of characteristics wherein it is transformed into ordinary differential equations [5]. These equations are the ODE-system of Eq. (1) itself and

$$\frac{\partial}{\partial\tau}p(\mathbf{x},\tau) = -p(\mathbf{x},\tau)\sum_{i=1}^{n}\frac{\partial}{\partial x_i}f_i(\mathbf{x},\tau). \tag{9}$$

Since the coefficient of the time derivative is const = 1, a distinction between two different time scales t and τ in an interval $\mathbb{T}\subseteq\mathbb{R}$ is not necessary. The solution of Eq. (9) is

$$p(\mathbf{x}|\mathbf{x}_0,t) = p_0(\mathbf{x}_0)\exp\left(\sum_{i=1}^{n}\frac{\partial}{\partial x_i}f_i(\mathbf{x},t)\,t\right) \tag{10}$$

with $p\colon \mathbb{R}^n\times D\times\mathbb{T}\to\mathbb{R}^+$. Therefore, the initial probability is dependent on the initial values during the exponential term increases with time.

For linear time-invariant systems an uncertain ensemble merges for $t\to\infty$ in a single sharp solution (the equilibrium point). The normalized probability density is volume-maintaining with respect to the $\mathbf{x}-p$-space $\mathcal{V}$, which means that in the course of movement the ensemble uncertainty decreases and the probability density increases.

For nonlinear systems, the derivatives of the exponential term of Eq. (10) may lead to a variation of the probability density. Therefore, three cases must be distinguished: (i) It is a Hamiltonian system and therefore divergenceless, (ii) it is a Hamiltonian system with canonical dissipative damping (CD-system) so that the divergence of the vector field $\mathbf{f}$ near the limit cycle is constant in $\mathbf{x}-\dot{\mathbf{x}}$-phase space $\mathcal{P}$ or (iii) it is a more general system in which the divergence is dependent on the state variables. Under the assumption of volume-maintaining in $\mathcal{V}$, the latter case has the property that for a negative divergence of $\mathbf{f}$ the volume in $\mathcal{P}$ increases and for a positive divergence decreases. It is easy to see that for an initial Dirac delta function (in the sense as the limit of a vanishing standard deviation of a normal distribution) the solution of (8) remains Dirac function in both cases [6].

A closed trajectory in $\mathcal{P}$ is called a limit cycle and corresponds to a fixed point on the Poincaré map. It is called stable if all neighbouring trajectories approach the limit cycle for $t\to\infty$. Therefore, for an ensemble of trajectories around the limit cycle, the transversal uncertainty goes against zero whereas the longitudinal uncertainty obeys Eq. (10). This means that the type of non-linearity substantially prescribes the behaviour near the limit cycle.

3. Examples

To illustrate the effects of a non-linearity $k(x,\dot{x})$ on the probability density $p(\mathbf{x},t)$, hereinafter some 2-dimensional examples with limit cycles are discussed. The second-order differential equations are in all cases of the form

$$\ddot{x} + k(x,\dot{x})\dot{x} + {\omega_0}^2 x = \text{const}\,. \tag{11}$$

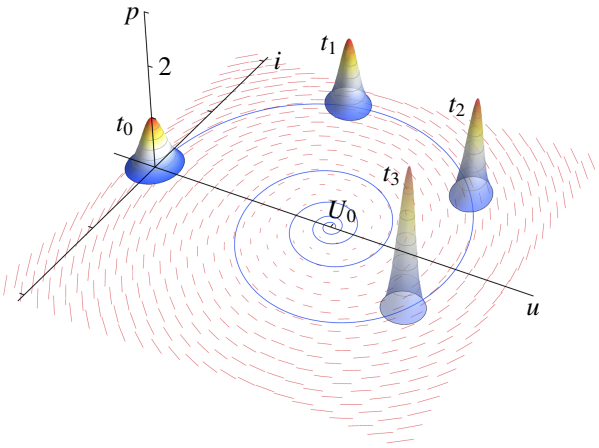

Figure 1: A Gaussian distribution with initial variances $c_u = 1.5\,\mathrm{V}^{-2}$ and $c_i = 10\,\mathrm{mA}^{-2}$ at $t = t_0$. The blue curve represents the trajectory of the distribution centre on the red curved vector field. For $t_{i+1} > t_i$ the ensemble uncertainty was attenuated. The initial variances go to zero for $t\to\infty$ so that the Gaussian distribution goes over into Dirac function.

3.1. Damped Resonant Circuit

As the first example, a damped resonant RCL-circuit with a constant voltage source U_0 and a nonlinear resistance $R(u,i)$ is given (see Fig. 2). The circuit will be closed at time $t = t_0$ so that the voltage $u(t)$ over the capacity C would go for a linear resistance R_0 and $t\to\infty$ to $u = U_0$ (equilibrium point). The differential equations read

$$C\frac{\mathrm{d}}{\mathrm{d}t}u = i, \tag{12a}$$

$$L\frac{\mathrm{d}}{\mathrm{d}t}i = U_0 - R(u,i)\,i - u. \tag{12b}$$

The Liouville equation is therefore

$$\frac{\partial}{\partial t}p + \frac{1}{C}i\frac{\partial}{\partial u}p + \frac{1}{L}\left(U_0 - R(u,i)\,i - u\right)\frac{\partial}{\partial i}p - \frac{p}{L}\frac{\partial}{\partial i}R(u,i)\,i = 0 \tag{13}$$

with $C, L\in\mathbb{R} > 0$. This partial differential equation can be solved by its characteristic differential equations which are the ODEs itself (Eq. 12) and

$$\frac{\partial}{\partial t}p = \frac{p}{L}\frac{\partial}{\partial i}R(u,i)\,i. \tag{14}$$

The solution of (14) is

$$p(u,i|u_0,i_0,t) = p_0(u_0,i_0)\exp\left(\frac{1}{L}\frac{\partial}{\partial i}R(u,i)\,i\,t\right) \tag{15}$$

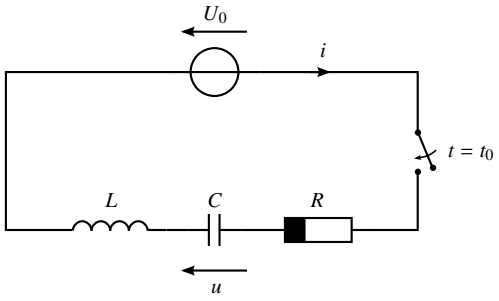

Figure 2: A damped resonant circuit with RCL in series: $R_0 = 22\,\Omega$, $C = 68\,\mu\text{F}$, $L = 470\,\text{mH}$ and $U_0 = 10\,\text{V}$.

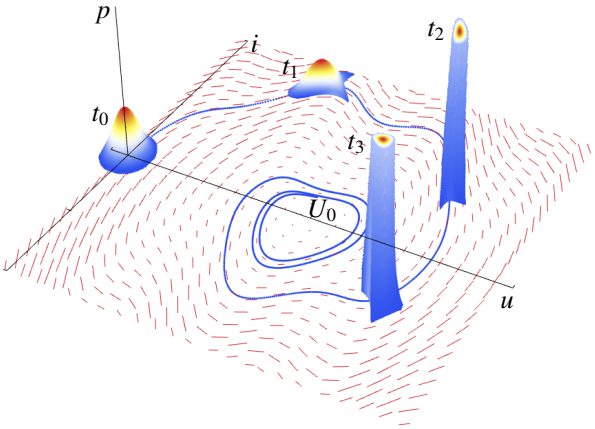

Figure 3: The same Gaussian initial probability as in Fig. 1, but the movement for a non-linearity. The distribution deforms non-isotropic and will run into a limit cycle around the equilibrium point U_0.

with $p\colon \mathbb{R}^2 \times D \times \mathbb{T} \to \mathbb{R}^+$ and $p_0\colon D \subseteq \mathbb{R}^2 \to \mathbb{R}$ as initial probability at $t = t_0$. In this example, p_0 is a Gaussian distribution around $\mathbf{s} = (u_s, i_s) \in D$ and therefore it is dependent on the initial values $u_0(u, i)$ and $i_0(u, i)$

$$p_0(u_0, i_0|u_s, i_s) = \exp\left(-c_u\,(u_0 - u_s)^2 - c_i\,(i_0 - i_s)^2\right) \quad (16)$$

with c_u and c_i as inverse variances. Accordingly, the variance is equivalent to the initial uncertainty. The vector $\mathbf{s}$ corresponds physically to the expected value of initial voltage u_0 and current i_0. Due to the non-linearity in Eq. (15) the probability is also voltage- and current-dependent. Since the nonlinear resistance only occurs in Eq. (12b), the probability density even depends on the current derivative of $R(u, i)$. Note, that the initial values are derived from the solutions of Eq. (12). They are, therefore, both dependent on the voltage u and current i. If the system is not analytically solvable, such for nonlinear systems, the ensemble dynamics can be given by a finite lattice around $\mathbf{s}$ whilst the resulting paths are summed according to Eq. (15).

For $R = \text{const}$ the damped resonant circuit is a linear time-invariant system and has an equilibrium point at $u = U_0$ with $i = 0$. This means that each initial probability with a finite volume in $\mathcal{V}$ must go to this equilibrium point. Therefore, an initial variance disappears for $t \to \infty$ and the probability distribution becomes Dirac function (see Fig. 1).

With nonlinear damping $R(u) = R_0\,(\alpha \sin(u) + \mu)$, the system dynamics is determined through bifurcation. With μ as bifurcation parameter the system forms for $\alpha \sin(U_0) < 0$ a limit cycle. Due to the longer path length the distribution is lagging behind the linear system and varies severely during it contracts non-isotropic (see Fig 3).

3.2. Canonical Dissipative Hamiltonian System: Rayleigh-Van der Pol

In Hamiltonian systems it is assumed that the energy is preserved. This property is indicated by the symplectic structure of Hamilton's equations [1]. However, this classical concept does not include the dissipative structure of real physical experiments. Therefore, the extension of Hamilton's equations with Canonical Dissipative Damping is an interesting possibility [3]. Let $\mathbf{S}$ be a symplectic $n \times n$ matrix and $g(H)$ an arbitrary function of Hamiltonian $H\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ then

$$\begin{pmatrix}\dot{\mathbf{x}}\\ \dot{\mathbf{y}}\end{pmatrix} = \mathbf{S}\begin{pmatrix}\partial_{\mathbf{x}}H\\ \partial_{\mathbf{y}}H\end{pmatrix} - g(H)\begin{pmatrix}\partial_{\mathbf{y}}H\\ \partial_{\mathbf{x}}H\end{pmatrix} \quad (17)$$

is a Canonical Dissipative Hamiltonian System.

The Rayleigh-Van der Pol system, which is a combination of the Rayleigh and Van der Pol system, satisfies this equation and contains a limit cycle. The equations of movement are

$$\begin{aligned}\dot{x} &= y,\\ \dot{y} &= \varepsilon\left(R^2 - x^2 - y^2\right)y - x\end{aligned} \quad (18)$$

with $\varepsilon \in \mathbb{R}^+$ as a parameter and $R \in \mathbb{R}^+$ as radius of the circular limit cycle. The Liouville equation reads

$$\begin{aligned}\dot{p} + y\frac{\partial}{\partial x}p + \left(\varepsilon\left(R^2 - x^2 - y^2\right)y - x\right)\frac{\partial}{\partial y}p\\ +\varepsilon\left(R^2 - x^2 - 3y^2\right)p = 0.\end{aligned} \quad (19)$$

In Fig. 4 it can be seen that after transition the shape of probability density near the limit cycle is nearly the same for two time points close to each other. For a linear transformation (axis scaling) $\sqrt{3}y \to \tilde{y}$ the solution of p is

$$p(x, \tilde{y}|x_0, \tilde{y}_0, t) = p_0(x_0, \tilde{y}_0)\exp\left(\varepsilon\left(x^2 + \tilde{y}^2 - R^2\right)t\right). \quad (20)$$

With a transformation into polar coordinates $(x, \tilde{y}) \to (r, \varphi)$ this solution has no coordinate dependent coefficients with respect to time. Therefore, the uncertainty of the Gaussian distribution near the limit cycle only depends on the initial probability p_0. If the variances ${c_x}^{-1}$ and ${c_{\tilde{y}}}^{-1}$ are great enough, so that the initial distribution overlapped with the limit cycle, even non-contiguous distributions may occur.

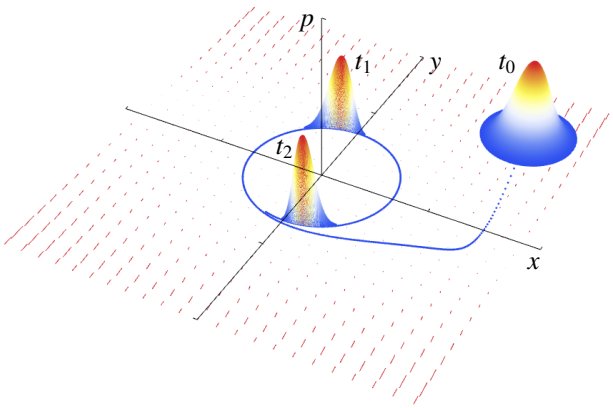

Figure 4: Rayleigh-Van der Pol system ($\varepsilon = 1$, $R = 1$, $c_x = c_y = 10$ and $x_s = y_s = 2$). After transition, the shape of probability density near the limit cycle for $t_1 < t_2$ is approximately the same.

3.3. Van der Pol Limit Cycle

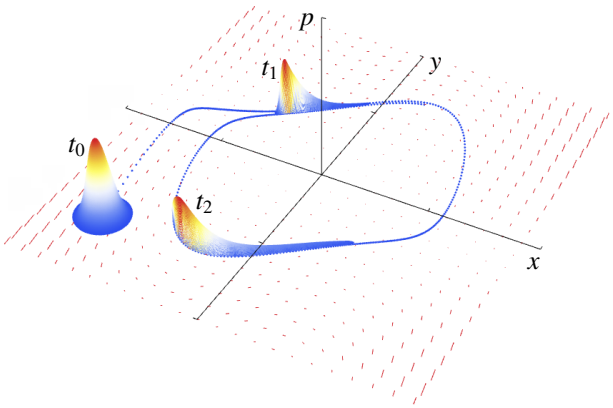

Figure 5: Van der Pol system ($\varepsilon = 1$, $R = 0.7$, $c_x = c_y = 30$ and $x_s = y_s = -2$). After transition, the shape of probability density near the limit cycle for $t_1 < t_2$ is different. The scaling in p-direction for each distribution is different because the maximum varies significantly.

The non-conservative Van der Pol system satisfies the conditions of Poincaré-Bendixson wherefore no chaos occur. The equations of movement are

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= \varepsilon \left(R^2 - x^2\right) y - x \end{aligned} \tag{21}$$

with $\varepsilon, R \in \mathbb{R}^+$. The Liouville equation differs from Eq. (19) only to the last term

$$\dot{p} + y\frac{\partial}{\partial x}p + \left(\varepsilon\left(R^2 - x^2\right)y - x\right)\frac{\partial}{\partial y}p + \varepsilon\left(R^2 - x^2\right)p = 0. \tag{22}$$

Near the limit cycle the uncertainty of the Gaussian distribution varies significantly and the ensemble moves at different speeds. In an "acceleration phase", the anterior parts are accelerated more strongly so that the distribution is stretched. Furthermore, it can be seen that the phase velocity of the maximum differs from the group velocity.

4. Conclusion

The generalized Liouville equation includes more information about the transient behaviour than the description by the state space equations since they are equivalent if the initial probability is a Dirac function. Moreover, the Liouville equation returns the probability density of each state vector in phase space.

If stable equilibrium points exists, all solutions that starts in the corresponding basin of attraction converge into it and the probability density becomes Dirac function for $t \to \infty$. For linear time-invariant systems the contraction of the initial probability is isotropic while in the nonlinear case the initial probability may be deformed.

If the system has a closed trajectory (limit cycle), it is crucial whether it is a canonical dissipative system or a more general. In the first case, the density does not change near the limit cycle because the energy of the system is preserved. In the latter, pumping and dissipation can be observed based on the time-evolution of the density function whereas *Erdmann, et al.* studied this effect based on a dissipation function [4].

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