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Delay-independent design of delay-coupled Bernoulli maps for inducing chaotic synchronization

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Abstract—The present paper investigates chaotic synchronization in delay-coupled Bernoulli maps. We provide a simple design of the map parameter and the coupling strength for inducing chaotic synchronization in situations where the number of maps, the detailed information of the network topology, and the connection delay are unknown. Even in such situations, a sufficient condition based on robust control theory leads to a simple design. Numerical simulations show the validity of our design.

1. Introduction

Synchronization, a typical nonlinear phenomenon that is observed in coupled oscillators, has been extensively investigated [1]. It is well known that the local stability of synchronization is equal to that of linear systems with time-variant parameters corresponding to the synchronized states [2]. In general, some numerical calculations are required in order to estimate the stability of synchronization.

An interaction between oscillators inevitably contains a connection delay due to finite signal-propagation speed. Thus, in recent years, delay-coupled oscillators have attracted much attention in nonlinear science [3, 4]. For delay-coupled oscillators, it is difficult to analyze the stability of synchronization. This is because the linear systems with time-variant parameters, which govern the stability of synchronization, have infinite-dimensional dynamics due to their time-delay terms.

Because delay-coupled Bernoulli maps have discrete-time dynamics and a constant slope, the stability of synchronization is equal to that of finite-dimensional linear systems without time-variant parameters. Thus, they can be easily analyzed; therefore, synchronization in delay-coupled Bernoulli maps has been investigated in several previous studies in various contexts, such as sublattice synchronization [5, 6], networks with multiple connection delays [7], the relation between a self-feedback delay and a connection delay [8], and unidirectional delay coupling [9].

Chaotic synchronization in delay-coupled Bernoulli maps can potentially be applied to create secure communication systems, since most of phenomena in these maps could be also observed in coupled chaotic semiconductor lasers [10–12]. Previous studies of delay-coupled Bernoulli maps provide the necessary and sufficient condition for chaotic synchronization under the assumption of a long delay limit. In practical situations, however, the connection delay is not always long. Thus, in such situations, these previous studies cannot guarantee the stability of chaotic synchronization.

The present study provides a delay-independent design for chaotic synchronization in delay-coupled Bernoulli maps. We consider practical situations where the connection delay, the number of maps, and detailed information about the network topology are unknown. A sufficient condition based on robust control theory allows us to obtain a simple design for the map parameter and the connection strength. We present some numerical simulations that confirm our design.

2. Delay-coupled maps

We now consider delay-coupled Bernoulli maps,

$$x_i(n+1) = f[x_i(n)] + \varepsilon u_i(n), \quad (i = 1, \dots, N), \tag{1}$$

where $x_i(n) \in \mathbb{R}$ is the state variable of the i-th map, $f(x) := (ax) \bmod 1$ is the nonlinear function of a Bernoulli map[1], and $\varepsilon \in [0, 1]$ denotes the coupling strength. The coupling signal is given by

$$u_i(n) = \frac{1}{d_i}\left\{\sum_{j=1}^{N} c_{ij} f[x_j(n-\tau)]\right\} - f[x_i(n)], \tag{2}$$

where $x_j(n-\tau)$ is the delayed-state variable of the j-th map, $\tau \in \mathbb{Z}^+$ is the connection delay, and c_{ij} represents the ij-th element of the adjacency matrix: if the i-th and j-th oscillators are coupled, then $c_{ij} = c_{ji} = 1$, otherwise, they equal 0. Thus, in order to simplify the stability analysis, the present study considers only bidirectional coupling. Self-feedback is forbidden (i.e., $c_{ii} = 0$). Here, $d_i := \sum_{j=1}^{N} c_{ij}$ denotes the number of maps connected to the i-th map. The network topology is governed by the matrix $\boldsymbol{C}$, whose elements are

[1] A Bernoulli map produces a sequence of independent identically distributed random variables; it has significant applications in digital communication systems [13].

given by $[\boldsymbol{C}]_{i,j} = c_{ij}/d_i$. It is well known that the eigenvalues of the matrix $\boldsymbol{C}$ always satisfy [7]

$$1 = \rho_0 \geq |\rho_1| \geq \cdots \geq |\rho_{N-1}|, \quad (3)$$

where $\rho_0 = 1$ corresponds to the eigenvector $[1 \cdots 1]^T$.

In the present paper, we will provide a design for the coupling strength ε and the map parameter a that will induce chaotic synchronization under the following specifications.

Specification (I): The upper limit on the eigenvalues of $\boldsymbol{C}$, $1 \geq \rho_{max} \geq |\rho_1|$, is given.

Specification (II): the number of maps N, detailed information about the network topology $[\boldsymbol{C}]_{i,j}$, and the connection delay τ are unknown.

These specifications are required for practical systems. This is because, in a huge network, not all of the information about the network (i.e., $\boldsymbol{C}$ and N) will be available, but a part of it (i.e., ρ_1) may be available; in addition, the connection delay will not be known.

3. Stability analysis

Since each of the rows in the connection matrix $\boldsymbol{C}$ sums to 1, the delay-coupled Bernoulli maps (1) and (2) have a synchronized manifold:

$$s(n) := x_1(n) = x_2(n) = \cdots = x_N(n). \quad (4)$$

By substituting Eq. (4) into Eqs. (1) and (2), we can see that $s(n)$ is governed by

$$s(n+1) = (1-\varepsilon)f[s(n)] + \varepsilon f[s(n-\tau)]. \quad (5)$$

Let us consider the dynamics around the synchronized manifold (4) by substituting the perturbation $\delta x_i(n) := x_i(n) - s(n)$ into Eqs. (1) and (2). Since the map has a constant slope (i.e., $\mathrm{d}f(x)/\mathrm{d}x|_{x=s(n)} = a$), the dynamics around the synchronized manifold (4) can be described by the following linear time-invariant system:

$$\delta\boldsymbol{x}(n+1) = (1-\varepsilon)a\delta\boldsymbol{x}(n) + \varepsilon a\boldsymbol{C}\delta\boldsymbol{x}(n-\tau), \quad (6)$$

where $\delta\boldsymbol{x}(n) := [\delta x_1(n)\ \delta x_2(n)\ \cdots\ \delta x_N(n)]^T$. The matrix $\boldsymbol{C}$ can be diagonalized by a transformation matrix $\boldsymbol{T}$ as $\boldsymbol{T}^{-1}\boldsymbol{C}\boldsymbol{T} = \mathrm{diag}(\rho_0, \ldots, \rho_{N-1})$. This diagonalization allows us to obtain

$$\boldsymbol{\xi}(n+1) = (1-\varepsilon)a\boldsymbol{\xi}(n) + \varepsilon a\,\mathrm{diag}(\rho_0, \ldots, \rho_{N-1})\boldsymbol{\xi}(n-\tau), \quad (7)$$

where $\boldsymbol{\xi}(n) := \boldsymbol{T}\delta\boldsymbol{x}(n)$. Thus, Eq. (7) can be divided into N modes, where mode q is given by

$$\xi_q(n+1) = (1-\varepsilon)a\xi_q(n) + \varepsilon a\rho_q\xi_q(n-\tau), \quad (q = 0, 1, \ldots, N-1). \quad (8)$$

Mode $q = 0$ in Eq. (8) denotes the time evolution of the synchronized manifold (i.e., (5)). Hence, for the stable synchronized manifold (4), the linear system (8) must be stable for all modes $q = 1, \ldots, N-1$; note that this does not include $q = 0$. The stability of each mode q is determined by the roots of the characteristic equation:

$$g(z, \rho_q) := z^{\tau+1} - (1-\varepsilon)az^{\tau} - \varepsilon a\rho_q. \quad (9)$$

Consequently, the synchronized manifold (4) is stable if and only if all the roots of Eq. (9) for $\forall q \in \{1, \ldots, N-1\}$ lie inside the unit circle.

4. Design of the map parameter and the coupling strength

For chaotic synchronization, the time evolution of the synchronized manifold (i.e., (5)) must behave chaotically. Since the dynamics of Eq. (5) are governed by the mode $q = 0$ in Eq. (8), the mode $q = 0$ must be unstable if chaotic synchronization is to be realized. Here, we provide the sufficient condition for the mode $q = 0$ to be unstable for any ε and τ.

Lemma 1. *Consider delay-coupled Bernoulli maps (1) and (2). The mode $q = 0$ in Eq. (8) is unstable for any coupling strength ε and any connection delay τ if*

$$a > 1. \quad (10)$$

Proof. From Eq. (9), the characteristic equation of the mode $q = 0$ is given by

$$g(z, 1) = z^{\tau+1} - (1-\varepsilon)az^{\tau} - \varepsilon a. \quad (11)$$

For real z, $\lim_{z\to+\infty} g(z, 1) > 0$ holds. Since $g(z, 1)$ defined in Eq. (11) is a continuous function, it has at least one real root greater than 1 on the real axis if $g(1, 1) = 1 - a$ is negative. Therefore, the mode $q = 0$ in Eq. (8) is unstable for any ε and τ if $g(1, 1) < 0 \Leftrightarrow a > 1$. □

Note that a previous study [7] provided an extended version of Lemma 1, in which multiple connection delays are considered. Now, we will provide the sufficient condition for Eq. (9) to be stable for all modes $q = 1, \ldots, N-1$. However, it is difficult to derive such a condition under the assumption that the connection delay τ is unknown, as stated in Specification (II). Hence, we will use the following lemma from robust control theory,

Lemma 2 ([14]). *An m-dimensional characteristic equation*

$$h(z) = z^m + \alpha_1 z^{m-1} + \cdots + \alpha_{m-1}z + \alpha_m \quad (12)$$

is stable if

$$1-\alpha_m^2 > |\alpha_1-\alpha_m\alpha_{m-1}|+|\alpha_2-\alpha_m\alpha_{m-2}|+\cdots+|\alpha_{m-1}-\alpha_m\alpha_1|. \quad (13)$$

Lemma 2 allows us to provide a sufficient condition for the characteristic equation (9) to be stable, independent of the connection delay τ. As a result, we obtain the following main theorem.

Theorem 1. *Let us consider delay-coupled Bernoulli maps (1) and (2). The time evolution of* $s(n)$ *is chaotic, and synchronized manifold (4) is stable, if Eq. (10) and*

$$(1-\varepsilon)a + \varepsilon a \rho_{max} < 1 \tag{14}$$

hold.

Proof. According to Lemma 1, the time evolution of $s(n)$ is chaotic if Eq. (10) holds. Thus, we will prove that the synchronized manifold (4) is stable (i.e., we will prove that Eq. (9) is stable for all modes $q = 1, \ldots, N-1$), if Eq. (14) holds.

If we take define the terms in Eq. (9) as follows,

$$\alpha_1 = -(1-\varepsilon)a, \quad \alpha_2 = \alpha_3 = \cdots = \alpha_{m-1} = 0,$$
$$\alpha_m = -\varepsilon a \rho_q, \quad m = \tau + 1, \tag{15}$$

then Eq. (13) can be rewritten as

$$(1-\varepsilon)a + \varepsilon a|\rho_q| < 1. \tag{16}$$

It should be noted that Eq. (16) is independent of τ. By means of the relation in Eq. (3), we see that the characteristic equation (9) is stable for all mode $q = 1, \ldots, N-1$, if Eq. (14) holds.

As a result, $s(n)$ is chaotic, and the synchronized manifold (4) is stable, if Eqs. (10) and (14) hold. □

A previous study [7] proved that the inequalities given by Eqs. (10) and (14) are necessary and sufficient conditions for chaotic synchronization only in the limit of a long delay (i.e., $\tau \to +\infty$). In contrast, the present study shows that these inequalities (10) and (14) are sufficient conditions for any connection delay (i.e.,$\forall \tau \in \{1, 2, \ldots\}$). Our results indicate that the parameter space of chaotic synchronization for the long-delay limit is included in the parameter space for any delay.

We see that the inequality given in Eq. (14) never holds if $\rho_{max} = 1$. Thus, Theorem 1 cannot guarantee the stability of chaotic synchronization for networks with $\rho_{max} = \rho_1 = 1$ (i.e., a bipartite graph).

From the above analytical results, we obtain the map parameter and the coupling strength.

Corollary 1. *Assume that* $\rho_{max} < 1$. *If the map parameter a is set to*

$$1 < a < \frac{1}{\rho_{max}}, \tag{17}$$

and the coupling strength is set to

$$\frac{a-1}{a(1-\rho_{max})} < \varepsilon \leq 1, \tag{18}$$

then Theorem 1 holds.

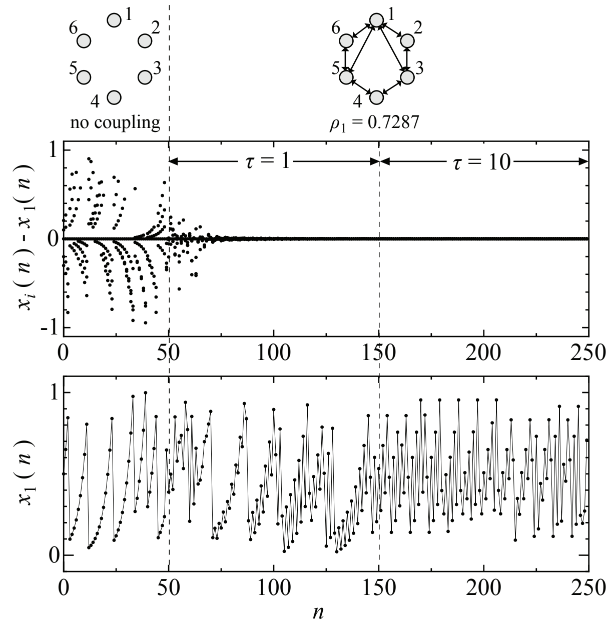

Figure 1: Time-series data of the delay-coupled Bernoulli maps ($N = 6$): all the maps are coupled at $n = 50$. For $n \in [50, 150)$ and $n \in [150, 250]$, the connection delays are fixed at $\tau = 1$ and $\tau = 10$, respectively. The map parameter is $a = 1.3$, and the coupling strength is $\varepsilon = 0.95$.

Let us briefly summarize this design. According to Specifications (I) and (II), only the parameter ρ_{max} is given. First, we check whether $\rho_{max} = 1$; if so, then we give up the design since Eq. (14) never holds. The map parameter a is then chosen from the range given in Eq. (17), and the coupling strength ε is chosen from the range given in Eq. (18). The designed parameter set (a, ε) guarantees the stability of chaotic synchronization for any N, any τ, and any network topology with $\rho_1 \leq \rho_{max}$.

5. Numerical simulations

This section numerically confirms the validity of our design, namely, Corollary 1. From Specification (I), $\rho_{max} = 0.75 < 1$ is given. According to the inequalities of Eqs. (17) and (18), the map parameter and the coupling strength are set to $a = 1.3$ and $\varepsilon = 0.95$, respectively. We will confirm whether the designed parameter set (a, ε) can successfully induce chaotic synchronization.

Figure 1 shows the time-series data of $x_i(n) - x_1(n)$ and $x_1(n)$ in the delay-coupled Bernoulli maps ($N = 6$). All the maps oscillate independently until $n = 50$. At $n = 50$, the maps are connected in a network with the eigenvalue $\rho_1 = 0.7287 < \rho_{max}$. For $n \in [50, 150)$ and $n \in [150, 250]$, the connection delay is fixed at $\tau = 1$ and $\tau = 10$, respectively [2].

[2] In order to investigate the local stability of the synchronized manifold, a small random noise with amplitude $[-1 \times 10^{-4}, 1 \times 10^{-4}]$ is added to all

Before coupling ($n < 50$), each isolated map oscillates chaotically. Thus, the values $x_i(n) - x_1(n)$ do not converge. After coupling ($n \geq 50$), $x_i(n) - x_1(n)$ converge to zero; this implies that the maps are synchronized. Even after changing τ, synchronization can be obtained. Furthermore, $x_1(n)$ oscillates chaotically regardless of the time. These results suggest that *chaotic* synchronization occurs in the maps. We see that the behavior of $x_1(n)$ in $n \in [50, 150)$ is totally different from that in $n \in [150, 250)$. This is due to the time evolution of the synchronized manifold (i.e., Eq. (5)), which depends on the connection delay τ. It can be concluded that the parameters designed according to Corollary 1 successfully induce chaotic synchronization.

6. Conclusion

This paper provides a simple design for inducing chaotic synchronization in delay-coupled Bernoulli maps. The design was derived by using robust control theory, although we considered severe situations in which the number of maps, detailed information about the topology, and the connection delay are unknown. The validity of our design was confirmed by numerical simulation.

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maps at each time step. The initial conditions are chosen to be $x_1(0) = 0.5001$, $x_2(0) = 0.2001$, $x_3(0) = 0.3001$, $x_4(0) = 0.6010$, $x_5(0) = 0.7070$, and $x_6(0) = 0.4703$.

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