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Sparsity structures for Koopman operators

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Abstract—We show that sparse structures – in form of subsystems – of a dynamical system induce decompositions of the Koopman and Perron-Frobenius operator. Functorial properties of these operators imply that eigenfunctions for the subsystems induce eigenfunctions for the whole system, and invariant measures for the whole system induce invariant measures of the subsystems. We reverse that result for invariant measures under a necessary compatibility condition. Further we demonstrate by a numerical example that exploitation of sparsity improves accuracy for extended dynamic mode decomposition. *

1. Summary

We assume to have a dynamical system $\dot{x} = f(x)$ with dynamics $f : \mathbb{R}^n \to \mathbb{R}^n$ and a set $X \subset \mathbb{R}^n$ that is positively invariant, i.e. solution starting in X stay in X for all positive times. The corresponding Koopman operator for $t \in \mathbb{R}_+$ is defined as $T_t g := g \circ \varphi_t$ for functions $g : X \to \mathbb{C}$ in a suitable function space, where φ_t denotes the solution map to the differential equation (see [1]). The operator T_t is linear, and so is its adjoint operator $P_t : M(X) \to M(X)$, the Perron-Frobenius operator, acting on the space of measures $M(X)$ by the pushforward $P_t = (\varphi_t)_\#$. We investigate how sparse structures in the dynamical system can be exploited for Koopman and Perron-Frobenius analysis.

The procedure we present is as follows: First, the subsystems need to be identified. The subsystems lead to a decomposition of the Koopman and Perron-Frobenius operator. Such a decomposition comes with the advantage of treating lower dimensional systems, which leads to reduction of computational complexity. Figure 1 illustrates this approach.

Definition 1. *For $I \subset \{1,\dots,n\}$ we call (I, f_I) a subsystem or a subsystem induced by I if $f_I := \Pi_I \circ f$ only depends on the states index by I where $\Pi_I : \mathbb{R}^n \to \mathbb{R}^{|I|}$ with $\Pi_I(x_1,\dots,x_n) = (x_i)_{i\in I}$ are projections in $\mathbb{R}^n$ onto canonical coordinates indexed by I by .*

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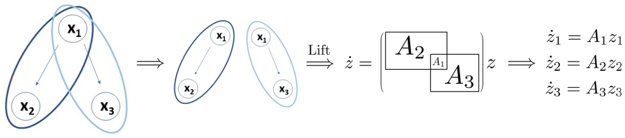

Figure 1: Illustration of sparse EDMD: 1. Identification of subsystems, 2. The subsystems induce a block structure for the Koopman operator (lift), 3. Exploitation of the block structure via decoupling of the subsystems.

The idea of a subsystem (I, f_I) is that we can treat it as a lower dimensional dynamical system, namely on $\mathbb{R}^{|I|}$ instead of $\mathbb{R}^n$. We view f_I as a (Lipschitz) vector field on $\mathbb{R}^{|I|}$. The semiflow induced by $f_I : \mathbb{R}^{|I|} \to \mathbb{R}^{|I|}$ is denoted $\varphi_t^I : \mathbb{R}^{|I|} \to \mathbb{R}^{|I|}$ for $t \in \mathbb{R}_+$ and, because (I, f_I) is a subsystem, it satisfies

$$\varphi_t^I \circ \Pi_I = \Pi_I \circ \varphi_t. \quad (1)$$

As a consequence, so-called eigenfunctions for the subsystems induce eigenfunctions of the whole system and eigenmeasures of the whole system induce eigenmeasures for the subsystems [2]. This result can be partially reversed, see [4]. In other words the subsystem allow for decompose the Koopman and Perron-Frobenius operator into the corresponding operators T_t^I and P_t^I on the subsystems while maintaining important spectral information.

Because the operators T_t^I and P_t^I correspond to lower dimensional spaces their computational complextiy is reduced. This can be exploited for extremal invariant measures computation from [3], and extended dynamic mode decomposition [4].

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