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Design of homoclinic bifurcation in PWC neuron model

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Abstract—In this study, a design procedure of a homoclinic bifurcation of oscillatory orbits in a piece-wise constant (ab. PWC) neuron model is presented. It is shown that, under an appropriate design, the PWC neuron model can exhibit homoclinic bifurcations such as a border-collision-type bifurcation similar to the blue-sky catastrophe.

1. Introduction

Many neuromorphic analog circuit models have been presented and their nonlinear dynamics have been investigated intensively (see the references in [1] and [2]). One of the most major neuromorphic analog circuit modeling approach is to utilize smooth nonlinearities of circuit elements as building blocks of smooth nonlinear functions that realize smooth nonlinear vector fields. On the other hand, non-smooth neuromorphic analog circuit models have been also presented (see the references in [1] and [2]). Among such non-smooth models, piece-wise constant (ab. PWC) models have relatively simple non-smooth vector fields. Our group has been developing PWC analog circuit neuron models inspired by PWC oscillators developed by the group of Tsubone, Matsuoka, and Saito [3][4], where advantages of such PWC analog circuit neuron models include easy to design qualitative vector field, easy to tune parameter values in implemented circuits, possibilities of theoretical analysis, and possibilities of theoretical design of neuron-like bifurcation scenarios. For example, our group has been developing PWC analog circuit neuron models that can mimic bifurcations of resting states (i.e., equilibrium points) of smooth ordinary differential equation neuron models. On the other hand, in this paper, a design procedure of homoclinic bifurcations of spiking states (i.e., oscillatory orbits) of the PWC neuron model is presented. One of the most attractive homoclinic bifurcation is the blue-sky catastrophe as illustrated in Fig. 1 [5]. In Fig. 1(a), a stable periodic orbit and an unstable periodic orbit (not illustrated) co-exist. In Fig. 1(b), a parameter value is changed and then the stable periodic orbit and the unstable periodic orbit approach. In Fig. 1(c), the parameter value is further changed, the stable periodic orbit and the unstable periodic orbit merge and vanish, and then a complicated bursting orbit suddenly (catastrophically) appears (from the blue-sky). It is known that smooth ordinary differential equation neuron models sometimes exhibit the blue-sky catastrophe [6]. In this paper, it is shown that, under an appropriate design, the PWC analog circuit neuron model can exhibit a border-collision-type bifurcation similar to the blue-sky catastrophe.

2. PWC neuron model

Fig. 2 shows a piece-wise constant (ab. PWC) analog circuit neuron model, which is a generalized version of our previous model [1][2]. Features of the PWC neuron model are summarized as follows.

- The capacitor voltages v and u correspond to a membrane potential and recovery variable of a neuron model, respectively.
- The current sources are voltage-controlled and have piece-wise constant characteristics, i.e., step-function-like characteristics.
- Due to the piece-wise constant voltage-controlled current sources, the model has a piece-wise constant vector field, which sometimes has a sliding vector field.
- The switch is voltage-controlled and realizes a firing reset of the membrane potential v to a reset level V_B.
- The voltage-controlled current source I_{bc} is newly introduced in this paper in order to realize homoclinic bifurcations of oscillatory orbits that cannot be observed in the previous model [1][2].

The dynamics of the PWC analog circuit neuron model for the case where the switch SW is opened is described by the following equations.

For $v < V_t(u)$ (switch SW is opened) :

$$C\frac{dv}{dt} = I_v(|v| + V_{in} - u),$$
$$C\frac{du}{dt} = I_u(av - u) + I_{bc}(v, u),$$

$$V_t(u) = \begin{cases} V_t^c & \text{if } u \geq Q, \\ \alpha u + \beta & \text{if } u < Q, \end{cases}$$

where the voltage-controlled current sources have the following PWC characteristics.

$$I_v(v_e) = \begin{cases} +I_{v+} & \text{if } v_e \geq 0, \\ -I_{v-} & \text{if } v_e < 0, \end{cases}$$

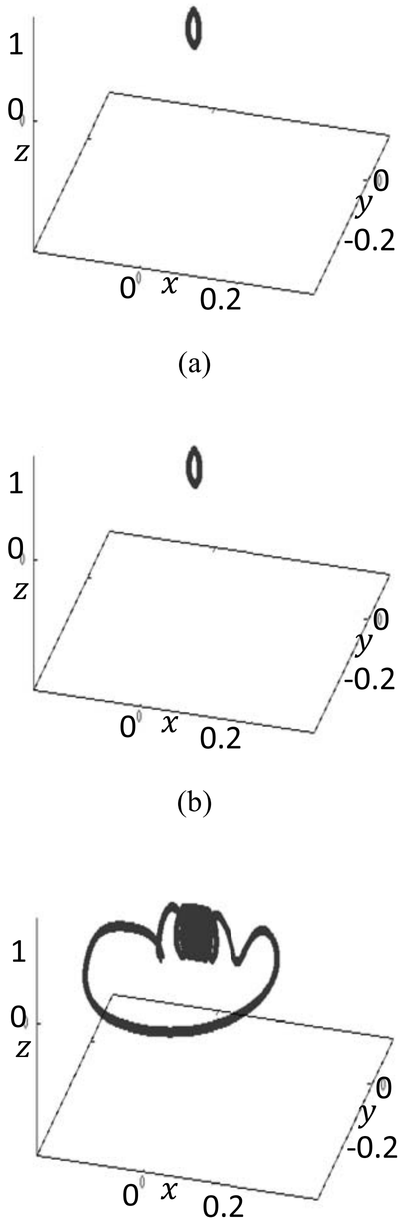

Figure 1: Blue-sky catastrophe from its typical example model [5]. (a) A stable periodic orbit and an unstable periodic orbit (not shown) co-exist. (b) As a parameter value is changed, the stable periodic orbit and the unstable periodic orbit approach. (c) The stable periodic orbit and the unstable periodic orbit merge and vanish. Then a complicated bursting orbit suddenly (catastrophically) appears (from the blue-sky).

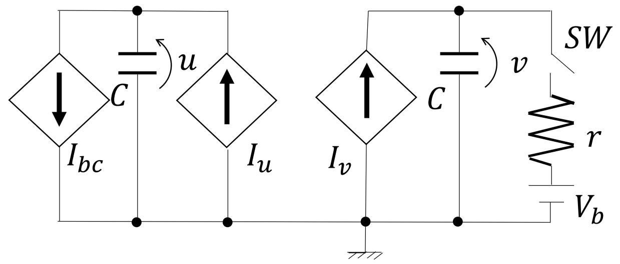

Figure 2: A piece-wise constant (ab. PWC) analog circuit neuron model which is a generalized version of our previous model [1][2].

$$I_v(v_e) = \begin{cases} +I_{u+} & \text{if } v_e \geq 0, \\ -I_{u-} & \text{if } v_e < 0, \end{cases}$$

$$I_{bc}(v,u) = \begin{cases} -I^c_{bc} & \text{if } \ v \geq V_b(u) \ \text{ and } \ Q \leq u \leq P, \\ 0 & \text{otherwise.} \end{cases}$$

The dynamics of the PWC analog circuit neuron model for the case where the switch SW is closed is described by the following equations.

$$\text{For } v = V_t(u) \quad (\text{switch } SW \text{ is closed}):$$

$$v \ \text{ is instantaneously reset to } \ V_b(u),$$

$$V_b(u) = \begin{cases} V^c_b & \text{if } u \geq Q, \\ \rho u + \sigma & \text{if } u < Q, \end{cases}$$

Fig. 3 shows time waveforms and phase plane trajectories of the PWC analog circuit neuron model. The phase plane in Fig. 3(a) explains the basic vector field of the model as follows. If $I^c_{bc} = 0$ (i.e., $I_{bc}(v,u)$ is always zero), then the PWC analog circuit neuron model has the following sub-threshold dynamics.

- If $|v| + V_{in} - u > 0$, $av - u > 0$, and $v < V_t(u)$, then v increases and u increases.
- If $|v| + V_{in} - u > 0$, $av - u < 0$, and $v < V_t(u)$, then v increases and u decreases.
- If $|v| + V_{in} - u < 0$, $av - u > 0$, and $v < V_t(u)$, then v decreases and u increases.
- If $|v| + V_{in} - u < 0$, $av - u < 0$, and $v < V_t(u)$, then v decreases and u decreases.

On the other hand, if I^c_{bc} is appropriately adjusted (i.e., I_{bc} is a voltage controlled current), then the PWC analog circuit neuron model additionally has the following sub-threshold dynamics.

- If $v \geq V_b(u)$, $Q \leq u \leq P$, and $v < V_t(u)$, then v increases and u decreases.

Note that if $I^c_{bc} = 0$, this case leads to the case where v increases and u increases. Furthermore, if v reaches $V_t(t)$, then the PWC analog circuit neuron model has the following super-threshold dynamics.

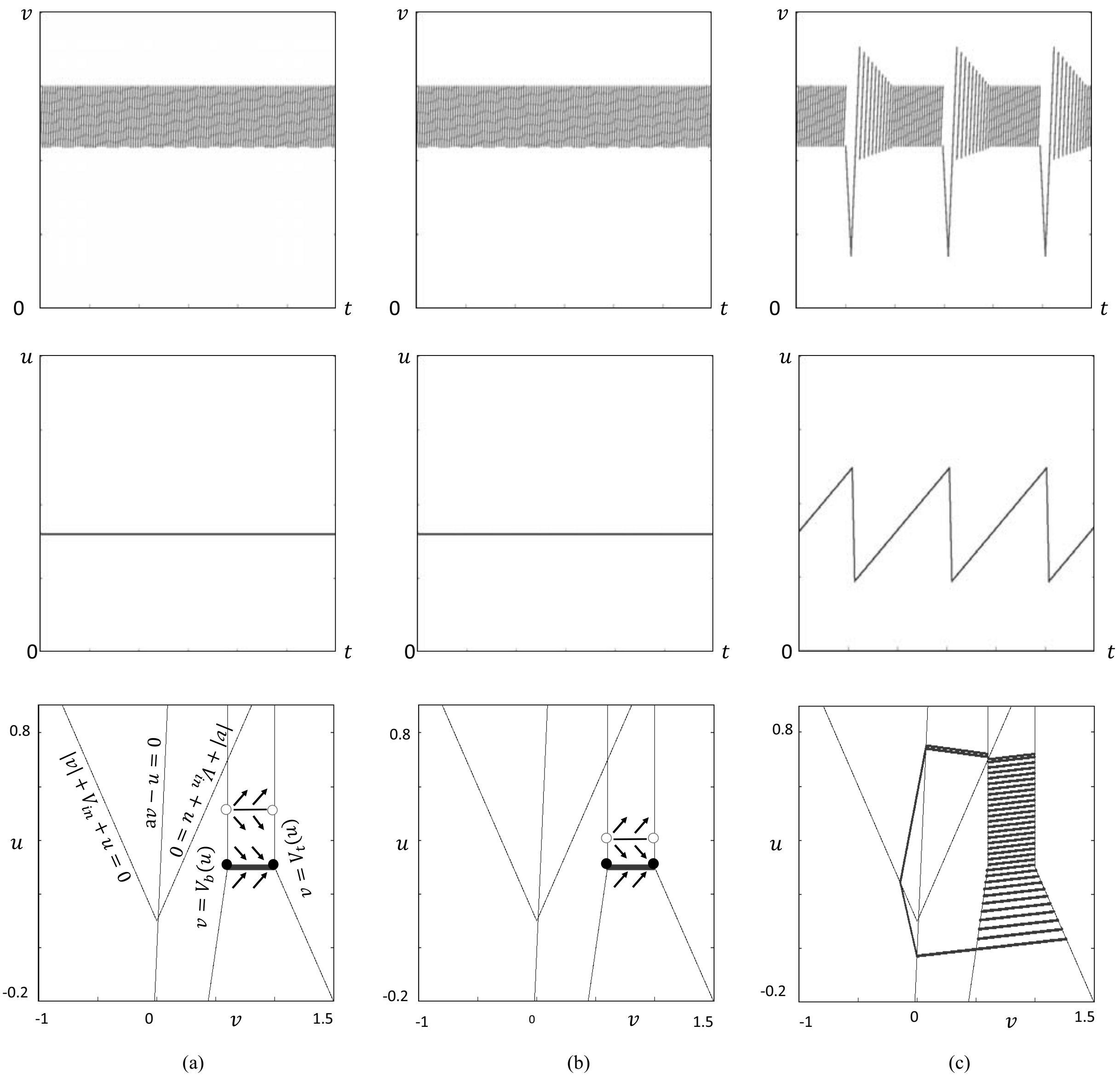

Figure 3: The voltage controlled current source I_v is characterized by $V_{in} = 0.1$, $I_{v+} = 0.16$, and $I_{v-} = 0.135$. The voltage controlled current source I_u is characterized by $a = 1$, $I_{u+} = 0.008$, and $I_{u-} = 0.31$. The threshold $V_t(u)$ is characterized by $\alpha = 0.3, \beta = 0.5$, and $V_t^c = 1$. The reset value $V_b(u)$ is characterized by $\rho = -1$, $\sigma = 1.3$, and $V_b^c = 0.6$. The voltage controlled current source I_{bc} is characterized by $Q = 0.3$. (a) $P = 0.5$. A stable periodic orbit (a pair of black dots) and an unstable periodic orbit (a pair of open dots) co-exist. (b) $P = 0.4$. As a parameter value is changed, the stable periodic orbit and the unstable periodic orbit approach. (c) $P = 0.3$. The stable periodic orbit and the unstable periodic orbit merge and vanish. Then a complicated bursting orbit suddenly (catastrophically) appears (from the blue-sky).

- If v reaches the threshold value $V_t(u)$, then v is instantaneously reset to the voltage $V_b(u)$.

3. BC bifurcation similar to Blue-Sky Catastrophe

Fig. 3 shows that the PWC analog circuit neuron model can exhibit a border-collision-type bifurcation similar to the blue-sky catastrophe as follows.

- In the case of Fig. 3(a), there exist a stable periodic orbit (ever bursting fast oscillation) illustrated by black dots and an unstable periodic orbit illustrated by open dots.
- In the case of Fig. 3(b), the value of the parameter P is changed. In this case, the stable periodic orbit and the unstable periodic orbit approach.
- In the case of Fig. 3(c), the value of the parameter P is further changed. In this case, the stable periodic orbit and the unstable periodic orbit merge and vanish. Then a complicated bursting orbit suddenly (catastrophically) appears (from the blue-sky).

Note that the situations in Fig. 3(a), (b), and (c) correspond to those in Fig. 1(a), (b), and (c), respectively. Note also that since the PWC analog circuit neuron model has the non-smooth PWC vector fields, every bifurcation is a border collision bifurcation [7]. Hence it can be concluded that the PWC analog circuit neuron model can exhibit a border-collision-type bifurcation similar to the blue-sky catastrophe.

4. Conclusion

In this paper, the design procedure of the voltage controlled current source I_{bc} so that the PWC neuron model exhibits a homoclinic bifurcation of oscillatory orbits was presented. It was shown that, under the appropriate design, the PWC analog circuit neuron model can exhibit homoclinic bifurcations such as the border-collision-type bifurcation similar to the blue-sky catastrophe. Future problems include: (a) theoretical analysis of the border-collision-type bifurcation similar to the blue-sky catastrophe in the PWC analog circuit neuron model, (b) design of other homoclinic bifurcations of the PWC analog circuit neuron model, and (c) hardware implementation of the PWC analog circuit neuron model.

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