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Predicting a parameter value at which a critical transition occurs from Lyapunov exponents in an estimated parameter space

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Abstract—This study estimates a parameter space from only two time-series data sets in order to predict a critical transition caused by a saddle-node bifurcation. By estimating the parameter space, we can plot a bifurcation diagram corresponding to the original bifurcation diagram. In addition, the Lyapunov exponent can also be approximated in the estimated parameter space and corresponds to the bifurcation diagram. Thereby, we expect that the parameter value at which the critical transition occurs is predicted. In numerical experiments, we estimate the parameter space for the coupled dynamics of water and vegetation, and we compare the bifurcation diagrams in the original and estimated parameter spaces. For predicting the critical transition, we confirm that the Lyapunov exponent reaches zero when the critical transition occurs.

1. Introduction

In 1994, Tokunaga *et al.* [1] proposed a method for reconstructing bifurcation diagrams (BDs), and their method was subsequently studied by several research groups [2]–[6]. By performing BD reconstruction, attractors can be estimated when bifurcation parameter values of a target system are changed. In related studies, Bagarinao *et al.* [7] and Itoh and Adachi [8] also reconstructed BDs using other methods. In particular, Itoh and Adachi were the first to use parameter space estimation to reconstruct a BD in which a saddle-node bifurcation occurs. Although all types of bifurcation are caused by changing parameter values, a saddle-node bifurcation is characterized by a sudden and considerable change in the associated state value.

Meanwhile, many researchers in recent years have focused on detecting critical transitions. In a critical transition, the associated state value changes suddenly and considerably, examples being a regime shift in an ecosystem, an asthma attack in medicine, and a systemic market crash in global finance, and some critical transitions have dynamics that are similar to those of a saddle-node bifurcation. Researchers have used early warning signals (EWSs) to detect critical transitions [9]. An EWS notifies a change of dynamics in a target system, and the general indicators of an EWS increase or decrease gradually as the parameter value approaches the value at which the critical transition occurs. However, although various research groups have shown the usefulness of EWSs for various problems, an EWS cannot predict when a critical transition will occur.

This study shows that the parameter value at which a critical transition occurs can be predicted by approximating the Lyapunov exponent in the estimated parameter space. This study defines the predicted parameter value as the parameter value at which the Lyapunov exponent reaches zero, because the Lyapunov exponent is zero when the bifurcation occurs. In addition, the target system is changed from the one-dimensional vegetation biomass model used in [8] to a multidimensional model [10] [11]. Thereby, this study also shows that the critical transition in the multidimensional model can be predicted by this method.

The rest of this paper is organized as follows. Section 2 introduces a method for estimating a parameter space using an extreme learning machine (ELM) and only time-series data sets. Section 3 introduces a method for approximating the Lyapunov exponent in the estimated parameter space. Section 4 shows results from numerical experiments. Finally, Section 5 gives conclusions.

2. Estimating Parameter Space Using an ELM

This section introduces a method for estimating a parameter space using an ELM and only two time-series data sets. The parameter space is estimated from time-series predictors trained to model the time-series data sets. It is assumed that the time-series data sets are generated from a system with different parameter values. This section begins by explaining the ELM as a time-series predictor. Then, this section explains the algorithm for parameter space estimation using the ELM [8].

2.1. Extreme Learning Machine

As proposed by Huang *et al.* in 2006 [12], an ELM is a neural network composed of three layers. The training targets in the ELM are the synaptic weights of the output neurons. The synaptic weights and biases of the hidden neurons are generated as random numbers and are not trained. The ELM is used as a time-series predictor in this study, because previous studies showed that it is useful for reconstructing BDs [3]–[6][8]. As in the numerical conditions for a previous system in which a saddle-node bifurcation

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occurs [8], the numbers of input and output neurons are set to one.

The ELM in this study is trained to output time-series data at time $t+1$ when time-series data are input at time t. To satisfy this, the ELM is represented as

$$\mathbf{h}(t) = g(\mathbf{w}x(t) + \mathbf{b}), \quad (1)$$
$$x(t+1) = \boldsymbol{\beta} \cdot \mathbf{h}(t), \quad (2)$$

where $x(t) \in \mathbb{R}$ and $x(t+1) \in \mathbb{R}$ are the state variables in the target system at t and $t+1$, respectively, $\mathbf{h}(t) \in \mathbb{R}^Y$ is the output vector of the hidden neurons, $g(\cdot)$ is a nonlinear function, $\mathbf{w} \in \mathbb{R}^Y$ and $\mathbf{b} \in \mathbb{R}^Y$ are the synaptic weights and the bias vectors, respectively, of the hidden neurons, and $\boldsymbol{\beta} \in \mathbb{R}^Y$ is the synaptic weight vector for the output neuron. Here, Y is the number of hidden neurons. In this study, we use the adjustable sigmoid function as follows:

$$g(\chi) = \frac{\epsilon_1}{1 + \exp(-\epsilon_3 \chi)} - \epsilon_2, \quad (3)$$

where ϵ_1 and ϵ_2 are the parameters to adjust the output range of the function, and ϵ_3 is the parameter to adjust the slope of the function.

The synaptic weight vector of the output neuron is trained by

$$\boldsymbol{\beta} = \mathbf{H}^{\dagger}\mathbf{d}, \quad (4)$$

where $\mathbf{H}^{\dagger}$ is the pseudo-inverse of the output matrix of hidden neurons $\mathbf{H} = [\mathbf{h}(1)\ \mathbf{h}(2)\ \cdots\ \mathbf{h}(L)]^T$, and $\mathbf{d} \in \mathbb{R}^L$ is the vector of desired output. Here, L is the length of the training data sets.

2.2. Parameter-space Estimation

This section introduces the algorithm for parameter space estimation using the ELM [8].

First, the synaptic weights of the output neuron in the ELM are trained to model a target system using two time-series data sets S_1 and S_2 by Sec. 2.1. Here, the synaptic weights and biases of the hidden neurons are randomly generated before modeling and are fixed during the parameter space estimation. Thereby, the predictor function $P(\cdot,\cdot)$ of the ELM is described as

$$x(t+1) = P\left(\boldsymbol{\beta}^{(n)}, x(t)\right), \ (n = 1, 2), \quad (5)$$

where $\boldsymbol{\beta}^{(n)} \in \mathbb{R}^Y$ is the trained synaptic weight vector for the time-series data set S_n.

Next, a parameter space is estimated from the two trained synaptic weight vectors. A differential synaptic weight vector is calculated from the trained synaptic weight vectors $\beta^{(1)}$ and $\beta^{(2)}$ by

$$\Delta\boldsymbol{\beta} = \boldsymbol{\beta}^{(2)} - \boldsymbol{\beta}^{(1)}. \quad (6)$$

The parameter space can be estimated from the differential synaptic weight vector. Specifically, a predictor with the estimated parameter space is given by

$$x(t+1) = P(\boldsymbol{\beta}, x(t)), \quad (7)$$
$$\boldsymbol{\beta} = \boldsymbol{\beta}^{(1)} + a\Delta\boldsymbol{\beta}, \quad (8)$$

where a is a bifurcation parameter in the estimated parameter space, and thereby we can generate a time-series equivalent to S_1 and S_2 when $a = 0$ and $a = 1$, respectively. Therefore, we expect to plot the BD in the estimated parameter space by generating time series from Eqs. (7) and (8) while changing the parameter a. In addition, the BD in the estimated parameter space whose range is $0 \le a \le 1$ corresponds to the BD in the original parameter space whose range is between p_1 and p_2, where p_1 and p_2 are parameters generated for time series S_1 and S_2, respectively. Note that a previous study [8] showed that the BD in the extrapolation region of the estimated parameter space corresponds to the original BD.

3. Approximating the Lyapunov Exponent in the Estimated Parameter Space

Because the Lyapunov exponent is zero when a bifurcation occurs, the parameter values at which the critical transition occurs can be predicted by approximating the Lyapunov exponent in the estimated parameter space. Based on [13]–[15], the Lyapunov exponent in the estimated parameter space is approximated from the results of applying the derivative of the nonlinear function $P(\cdot,\cdot)$ in Eq. (7) as follows:

$$\mu = \frac{1}{\varphi} \sum_{t=1}^{\varphi} \frac{dP(\boldsymbol{\beta}, x(t))}{dx(t)}, \quad (9)$$

where φ is the number of trials.

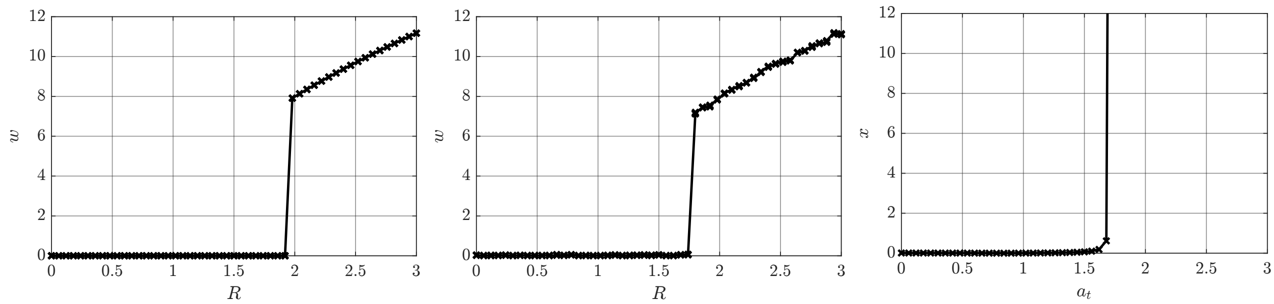
4. Numerical Experiments

This section shows results for comparing the BDs in the original and estimated parameter spaces and predicting the parameter value at which the critical transition occurs from the Lyapunov exponent. The coupled dynamics of water and vegetation [10] [11] are used as the target system in this numerical experiment. Here, the target system is changed from the one-dimensional vegetation biomass model used in [8] to a multidimensional model [10] [11]. Thereby, this study also shows that the critical transition in the multidimensional model can be predicted.

4.1. Experimental Conditions

The coupled dynamics of water and vegetation [10] [11] are

$$\frac{dw}{dt} = R - \alpha w - \lambda w B + v_1, \quad (10)$$
$$\frac{dB}{dt} = \rho w B\left(1 - \frac{B}{wB_c}\right) - \mu\frac{B}{B + B_0} + v_2, \quad (11)$$
$$(12)$$



(a) Original parameter space without white Gaussian noise.
(b) Original parameter space with white Gaussian noise.
(c) Estimated parameter space.

Figure 1: Bifurcation diagrams in the original and estimated parameter spaces.

Where α is the rate of soil water loss, λ is the consumption rate of water by the biomass, R is the rainfall rate, ρ is the maximum biomass growth rate, B_c is the carrying capacity of the biomass, μ is the maximum grazing rate, B_0 is the biomass amount at which the grazing rate is half its maximum value , and v_1 and v_2 correspond to external noise. The bifurcation parameter of this dynamic system is R. Therefore, the other parameters are set to $\alpha = 1.0$, $\lambda = 0.12$, $\rho = 1$, $B_c = 10$, $\mu = 2$, and $B_0 = 1$. Under these conditions, the critical transition occurs when the bifurcation parameter R is between 1 and 2. The parameter value at which the critical transition occurs is changed by the influence of noise from the characteristics of the saddle-node bifurcation.

For the numerical experiments, the time-series data sets were generated using a fourth-order Runge–Kutta method in which the time increment was 0.1. For training data, the two time-series data sets were generated with the bifurcation parameter values of $R = 1.0$ and $R = 1.2$. Here, the length of each time series was 5000.

The numbers of input, hidden, and output neurons in the ELM were 1, 4, and 1, respectively. The parameters ϵ_1, ϵ_2, and ϵ_3 of the adjustable sigmoid function were set to 10, 0, and 0.001, respectively. In this study, two sequences of white Gaussian noise v_1 and v_2 were used, with their mean and variance set to 0.0 and 0.01, respectively.

4.2. Bifurcation Diagrams in Original and Estimated Parameter Spaces

Figures 1(a) and (b) show the BDs without and with white Gaussian noise in the original parameter space, and Fig. 1(c) shows the BD in the estimated parameter space. Here, the mean and variance of the white Gaussian noise in Fig. 1(b) are 0.0 and 0.01, respectively, and thus the noise conditions are the same as those for the time-series generation. We see that the presence of noise changes the parameter value at which the critical transition occurs. To compare the BDs in the original and estimated parameter spaces, the parameter in the estimated parameter space is converted by

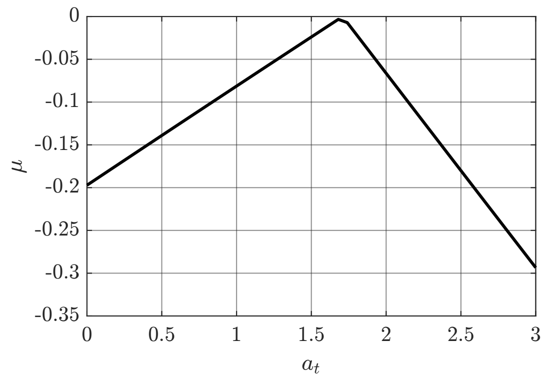


Figure 2: Lyapunov exponent in estimated parameter space.

$$a_t = 1.0 + a(1.2 - 1.0) \tag{13}$$

to correspond with the original parameter.

Comparing the BDs in the original and estimated parameter spaces, we see that the critical transition can be predicted from only time-series data sets before the critical transition occurs. On the other hand, we see that the critical transition does not occur when the parameter value is decreased from $a_t = 1$. The predicted parameter value at which the critical transition occurs is around 1.6, which is close to that in the BD in the original parameter space with noise.

4.3. Lyapunov Exponent in Estimated Parameter Space

The Lyapunov exponent in the estimated parameter space is shown in Fig. 2. Comparing the Lyapunov exponent to the BD in Fig. 1(c), the parameter value at which the Lyapunov exponent is close to zero corresponds to the parameter value at which the critical transition occurs. Thereby, the critical transition can be predicted by approximating the Lyapunov exponent in the estimated parameter space.

5. Conclusion

This study showed that a critical transition can be predicted by estimating the parameter space from only two time-series data sets before the critical transition occurs. We demonstrated the parameter space estimation for the coupled dynamics of water and vegetation. Comparing the BDs in the original and estimated parameter spaces, we saw that the parameter values at which the critical transition occurred were almost the same. In addition, the parameter value at which the Lyapunov exponent was close to zero corresponded to the parameter value at which the critical transition occurred in the BD. Thereby, we saw that the parameter value at which the critical transition occurred could also be predicted by approximating the Lyapunov exponent in the estimated parameter space.

This study has shown that the parameter value at which a critical transition occurs can be predicted when the parameter space is estimated with high precision. However, the estimation accuracy is sometimes low even if the parameter space is estimated under the same conditions. To estimate the parameter space with consistently high accuracy, ELMs used in numerical experiments under various conditions will be analyzed.

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References

[1] R. Tokunaga, S. Kajiwara and S. Matsumoto, "Reconstructing bifurcation diagrams only from time-waveforms," *Physica D*, vol.79, pp.348–360, 1994.

[2] G. Langer and U. Parlitz, "Modeling parameter dependence from time series," *Physical Review E*, vol.70, 2004.

[3] Y. Itoh, Y. Tada and M. Adachi, "Reconstructing bifurcation diagrams with Lyapunov exponents from only time-series data using an extreme learning machine," *Nonlinear Theory and Its Applications, IEICE*, vol.8, no. 1, pp. 2–14, 2017.

[4] Y. Itoh and M. Adachi, "Tracking bifurcation curves in the Henon map from only time-series datasets," *Nonlinear Theory and Its Applications, IEICE*, vol.2, no.3, pp.1101–1111, 2019.

[5] Y. Itoh, S. Uenohara, M. Adachi, T. Morie, and K. Aihara, "Reconstructing bifurcation diagrams only from time-series data generated by electronic circuits in discrete-time dynamical systems," *chaos*, vol.30, pp.013128-1–11, 2020.

[6] Y. Itoh and M. Adachi, "Reconstructing one- and two-bifurcation diagrams of all components in the Rossler equations only from time-series data sets," *Nonlinear Theory and Its Applications, IEICE*, vol.12, no.3, pp.391–398, 2021.

[7] E. Bagarinao, K. Pakdaman, T. Nomura and S. Sato, "Reconstructing bifurcation diagrams from noisy time series using nonlinear autoregressive models," *Physical Review E*, vol.60, no. 1, 1999.

[8] Y. Itoh and M. Adachi, "Bifurcation diagrams in estimated parameter space using a pruned extreme learning machine," *Physical Review E*, vol.98, pp.013301-1–12, 2018.

[9] M. Scheffer *et al.* , "Early-warning signals for critical transitions," *Nature*, vol.461, pp.53–59, 2009.

[10] V. Guttal and C. Jayaprakash, "Impact of noise on bistable ecological systems," *Ecological Modeling*, vol.201, pp.420–428, 2007.

[11] V. Guttal and C. Jayaprakash, "Changing skewness: an early warning signal of regime shifts in ecosystems," *Ecology Letters*, vol.11, pp.450–460, 2008.

[12] G. B. Huang, Q. Y. Zhu and C. K. Siew, "Extreme Learning Machine: Theory and Applications," *Neurocomputing*, vol.70, pp.489–501, 2006.

[13] I. Shimada and T. Nagashima, "A numerical approach to ergodic problem of dissipative dynamical systems," *Prog. Theor. Phys.*, vol.61, no.6, pp.1605–1616, 1979.

[14] M. Sano and Y. Sawada, "Measurement of the Lyapunov Spectrum from Chaotic Time Series," *Phys. Rev. Lett.*, vol.55, 1985.

[15] M. Adachi and M. Kotani, "Identification of Chaotic Dynamical Systems with Back-Propagation Neural Networks," *IEICE Trans. Fundamentals*, vol.E77-A, no.1, pp.324–334, 1994.