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Complete Multipartite Graphs Maximize Algebraic Connectivity in the Neighborhood based on 2-Switch

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Abstract—The second smallest eigenvalue of the Laplacian matrix, also known as the algebraic connectivity, is an important quantity for various network systems because it indicates how well the network is connected. The algebraic connectivity also characterizes some dynamic processes on networks such as consensus algorithms for multi-agent networks. In this paper, we prove that the algebraic connectivity of any complete multipartite graph is not less than that of all graphs obtained from it by applying a 2-switch. This is a generalization of the authors' previous result about complete bipartite graphs.

1. Introduction

The algebraic connectivity [1], which is defined as the second smallest eigenvalue of the Laplacian matrix, indicates how well the network is connected. The algebraic connectivity also characterizes some dynamic processes on multi-agent networks. For example, the convergence rate of the average consensus algorithm proposed by Olfati-Saber and Murray [2] is determined by the algebraic connectivity of the communication graph. Therefore, finding graphs with a high algebraic connectivity under certain constraints on the topology is an important problem from both a theoretical and a practical point of view.

Ogiwara *et al.* [3, 4] considered the problem of finding graphs that maximize or locally maximize the algebraic connectivity in the space of graphs with a fixed number of vertices and edges. They proved that some well-known classes such as star graphs, cycle graphs, complete bipartite graphs maximize the algebraic connectivity under certain conditions. They also proved that cycle graphs, complete bipartite graphs, and circulant graphs locally maximize the algebraic connectivity.

Recently, the authors of the present paper studied the problem of finding graphs that maximize or locally maximize the algebraic connectivity in the space of graphs with a fixed degree sequence [5]. This is closely related to the problem in which a communication topology has to be found for the fastest or nearly fastest consensus when the number of communication channels for each agent is fixed. They proved that any complete bipartite graph composed of six vertices or more is a local maximizer of the algebraic connectivity in the sense that it has the largest algebraic connectivity among all graphs in the neighborhood, where the neighborhood of a graph G is defined as the set of all graphs that can be obtained from G by applying a 2-switch. A 2-switch is a well-known graph transformation that does not change the degree sequence [6].

In this paper, as a generalization of the authors' previous result [5], we prove that any complete multipartite graph is a local maximizer of the algebraic connectivity in the same sense as above. We first review definitions of the algebraic connectivity maximizing (ACM) graph and the algebraic connectivity locally maximizing (ACLM) graph. We next study some properties of eigenvalues and eigenvectors of the Laplacian matrix of the complete multipartite graph. We finally present the main results of this paper.

2. Algebraic Connectivity Locally Maximizing Graph

Throughout this paper, by a graph, we mean a simple undirected graph. Let $G = (V(G), E(G))$ be a graph composed of n vertices and m edges, where $V(G) = \{1, 2, \ldots, n\}$ is the vertex set and $E(G) = \{e_1, e_2, \ldots, e_m\}$ is the edge set. Each edge is expressed as an unordered pair of two distinct vertices like $\{i, j\}$. The Laplacian matrix of G is defined by

$$L(G) = D(G) - A(G)$$

where $A(G) = (a_{ij}(G))$ is the adjacency matrix defined by

$$a_{ij}(G) = \begin{cases} 1, & \text{if } \{i, j\} \in E(G), \\ 0, & \text{otherwise,} \end{cases}$$

and $D(G)$ is the degree matrix defined by

$$d_i(G) = |\{j \mid \{i, j\} \in E(G)\}| .$$

Because $L(G)$ is always positive semi-definite, its eigenvalues are real and nonnegative. So we hereafter denote them by $\lambda_1(G) \le \lambda_2(G) \le \cdots \le \lambda_n(G)$. Furthermore, because $L(G)\mathbf{1} = \mathbf{0}$ holds, where $\mathbf{1}$ and $\mathbf{0}$ are the vectors of all ones and all zeros, respectively, the smallest eigenvalue of $L(G)$ is always zero, that is, $\lambda_1(G) = 0$.

The algebraic connectivity [1] is defined as follows.

Definition 1 The second smallest eigenvalue $\lambda_2(G)$ of $L(G)$ is called the algebraic connectivity of the graph G.

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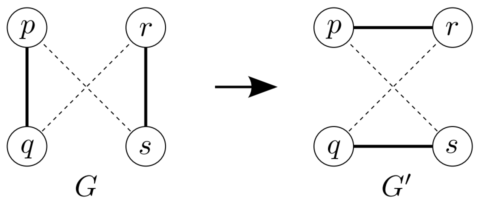

Figure 1: 2-switch. Each dotted line means that there may or may not exist an edge.

Suppose that a graph $G = (V(G), E(G))$ has four distinct vertices such that $E(G)$ contains $\{p, q\}$ and $\{r, s\}$ but neither $\{i, k\}$ nor $\{j, l\}$. Let $G' = (V(G'), E(G'))$ be the graph obtained from G by removing two edges $\{p, q\}$ and $\{r, s\}$, and by adding two new edges $\{p, r\}$ and $\{q, s\}$ (see Fig. 1). This transformation is called 2-switch. It is clear that the degree matrix does not change before and after the application of a 2-switch. Moreover, it is well known that, for any pair of graphs G and G' such that $D(G) = D(G')$, G can be transformed into G' by applying 2-switches sequentially [6].

We now present definitions of algebraic connectivity maximizing graphs and algebraic connectivity locally maximizing graphs, both of which were first introduced in [5].

Definition 2 A graph G is called an algebraic connectivity maximizing (ACM) graph in $\mathcal{G}_{D(G)}$ if

$$\forall G' \in \mathcal{G}_{D(G)}, \quad \lambda_2(G) \geq \lambda_2(G'),$$

where $\mathcal{G}_{D(G)}$ is the set of all graphs having the same degree matrix as G.

Definition 3 A graph G is called an algebraic connectivity locally maximizing (ACLM) graph in $\mathcal{G}_{D(G)}$ if

$$\forall G' \in \mathcal{N}_{D(G)}(G), \quad \lambda_2(G) \geq \lambda_2(G'),$$

where $\mathcal{G}_{D(G)}$ is same as Definition 2, and $\mathcal{N}_{D(G)}(G)$ is the set of all graphs obtained from G by applying a single 2-switch.

It is apparent from these definitions that if G is an ACM graph in $\mathcal{G}_{D(G)}$ then G is an ACLM graph in $\mathcal{G}_{D(G)}$. However, the converse is not true.

The following lemma shows that in some special cases G can be proved to be an ACM graph in $\mathcal{G}_{D(G)}$ by examining only graphs in $\mathcal{N}_{D(G)}(G)$.

Lemma 1 If $\mathcal{N}_{D(G)}(G) = \emptyset$ or all graphs in $\mathcal{N}_{D(G)}(G)$ are isomorphic to G then G is an ACM graph in $\mathcal{G}_{D(G)}$.

Proof: We first consider the case where $\mathcal{N}_{D(G)}(G) = \emptyset$. Suppose that $\mathcal{G}_{D(G)}$ contains a graph G' that is not G. Then G can be transformed into G' by applying 2-switches sequentially, and hence $\mathcal{N}_{D(G)}(G)$ must contain at least one graph. However, this contradicts $\mathcal{N}_{D(G)}(G) = \emptyset$. Therefore, we conclude that $\mathcal{G}_{D(G)} = \{G\}$ and hence G is the ACM graph in $\mathcal{G}_{D(G)}$. We next consider the case where all graphs in $\mathcal{N}_{D(G)}(G)$ are isomorphic to G. Suppose that there exists a $G' \in \mathcal{G}_{D(G)}$ that is not isomorphic to G. Let $G_0(= G), G_1, \ldots, G_{k-1}, G_k(= G')$ be a sequence of graphs such that G_{i+1} is obtained from G_i by a 2-switch. Then there exists an $i\,(\geq 2)$ such that G_i is isomorphic to G_1 but G_{i+1} is not. However, this contradicts the assumption that all graphs in $\mathcal{N}_{D(G)}(G)$ are isomorphic to G. Therefore, we conclude that all graphs in $\mathcal{G}_{D(G)}$ are isomorphic to G and hence G is an ACM graph in $\mathcal{G}_{D(G)}$. □

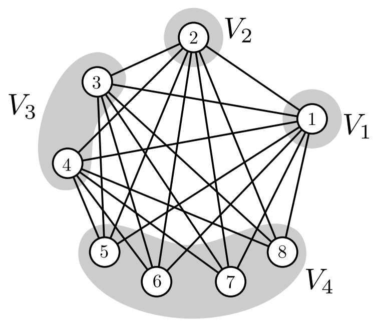

Figure 2: Complete 4-partite graph $K_{1,1,2,4}$. The algebraic connectivity of this graph is 4.

3. Complete Multipartite Graphs are Algebraic Connectivity Locally Maximizing Graphs

If the vertex set $V(G)$ of a graph G is partitioned into $k\,(\geq 2)$ disjoint nonempty subsets $V_1, V_2, \ldots, V_k$ such that two vertices i and j are adjacent if and only if they belong to different subsets, then G is called a complete k-partite graph and denoted by $K_{n_1,n_2,\ldots,n_k}$ where $n_l = |V_l|$ for $l = 1, 2, \ldots, k$. A complete multipartite graph is shown in Fig.2. In the following, we assume without loss of generality that $n_1 \leq n_2 \leq \cdots \leq n_k$. Also, we denote the index of the subset to which vertex i belongs by $\pi(i)$, that is, $i \in V_{\pi(i)}$ for $i = 1, 2, \ldots, n\,(= \sum_{l=1}^{k} n_l)$. Then the Laplacian matrix $L(K_{n_1,n_2,\ldots,n_k})$ is given by

$$L(K_{n_1,n_2,\ldots,n_k})_{ij} = \begin{cases} n - n_{\pi(i)}, & \text{if } i = j, \\ 0, & \text{if } \pi(i) = \pi(j) \text{ and } i \neq j, \\ -1, & \text{if } \pi(i) \neq \pi(j). \end{cases}$$

It is well known that eigenvalues of the Laplacian matrix of $K_{n_1,n_2,\ldots,n_k}$ are $0, n - n_k, n - n_{k-1}, \ldots, n - n_1, n$ and their multiplicities are $1, n_k - 1, n_{k-1} - 1, \ldots, n_1 - 1, k - 1$, respectively. In this section, we show that any complete k-partite graph $K_{n_1,n_2,\ldots,n_k}$ is an ACLM graph in $\mathcal{G}_{D(K_{n_1,n_2,\ldots,n_k})}$.

Lemma 2 Suppose we are given a complete multipartite graph $K_{n_1,n_2,\ldots,n_k}$. Let $\boldsymbol{v} = (v_1, v_2, \ldots, v_n)^{\mathrm{T}}$ be any vector such that i) $\|\boldsymbol{v}\| \neq 0$, ii) $v_i = 0$ for all $i \notin V_k$, and iii) $\boldsymbol{v}^{\mathrm{T}}\mathbf{1} = 0$. Then $\boldsymbol{v}$ is an eigenvector of $L(K_{n_1,n_2,\ldots,n_k})$ associated with $\lambda_2(K_{n_1,n_2,\ldots,n_k}) = n - n_k$.

Proof: Let $\boldsymbol{v}$ be any vector satisfying the three conditions. Then, for any i such that $i \notin V_k$ (or $\pi(i) \neq k$), we have

$$(L(K_{n_1,n_2,\ldots,n_k})\boldsymbol{v})_i = \sum_{j \in V_k} L(K_{n_1,n_2,\ldots,n_k})_{ij} v_j = -\sum_{j \in V_k} v_j = 0 = (n - n_k) v_i.$$

Also, for any i such that $i \in V_k$ (or $\pi(i) = k$), we have

$$(L(K_{n_1,n_2,\ldots,n_k})\boldsymbol{v})_i = \sum_{j \in V_k} L(K_{n_1,n_2,\ldots,n_k})_{ij} v_j = (n - n_k) v_i.$$

Therefore, $\boldsymbol{v}$ satisfies $L(K_{n_1,n_2,\ldots,n_k})\boldsymbol{v} = (n - n_k)\boldsymbol{v}$ which completes the proof. □

Lemma 3 If $n_{k-1} = 1$, $\mathcal{N}_{D(K_{n_1,n_2,\ldots,n_k})}(K_{n_1,n_2,\ldots,n_k})$ is the empty set. If $n_{k-1} = n_k = 2$, all graphs in $\mathcal{N}_{D(K_{n_1,n_2,\ldots,n_k})}(K_{n_1,n_2,\ldots,n_k})$ are isomorphic to $K_{n_1,n_2,\ldots,n_k}$.

Proof: Suppose first that $n_k = 1$. Then $K_{n_1,n_2,\ldots,n_k}$ is a complete graph, and hence $\mathcal{N}_{D(K_{n_1,n_2,\ldots,n_k})}(K_{n_1,n_2,\ldots,n_k}) = \emptyset$. Suppose next that $n_{k-1} = 1$ and $n_k \geq 2$. In this case, because all pairs of vertices, except pairs of vertices in V_k, are adjacent, there are no four vertices p, q, r, s such that $E(K_{n_1,n_2,\ldots,n_k})$ contains both $\{p, q\}$ and $\{r, s\}$ but neither $\{p, r\}$ nor $\{q, s\}$. Suppose finally that $n_{k-1} = n_k = 2$. In this case, there exist four vertices p, q, r, s such that $E(K_{n_1,n_2,\ldots,n_k})$ contains both $\{p, q\}$ and $\{r, s\}$ but neither $\{p, r\}$ nor $\{q, s\}$. We assume without loss of generality that $V_{k-1} = \{p, r\}$ and $V_k = \{q, s\}$. Then p, q, r, s are adjacent to all vertices except r, s, p, q, respectively. Let G be the graph obtained from $K_{n_1,n_2,\ldots,n_k}$ by applying a 2-switch to these four vertices as shown in Fig.1. Then, in G, vertices p, q, r, s are adjacent to all vertices except q, p, s, r, respectively. If we relabel vertices in G as $q \to r$ and $r \to q$, the resulting graph is identical to $K_{n_1,n_2,\ldots,n_k}$. Therefore, G is isomorphic to $K_{n_1,n_2,\ldots,n_k}$. □

From Lemmas 1 and 3, we have the following result.

Corollary 1 If $n_{k-1} = 1$ or $n_{k-1} = n_k = 2$, $K_{n_1,n_2,\ldots,n_k}$ is an ACM graph in $\mathcal{G}_{D(K_{n_1,n_2,\ldots,n_k})}$.

Now we give the first main theorem of this paper.

Theorem 1 Any $K_{n_1,n_2,\ldots,n_k}$ with $n_k \geq 4$ is an ACLM graph in $\mathcal{G}_{D(K_{n_1,n_2,\ldots,n_k})}$.

Proof: If $\mathcal{N}_{D(K_{n_1,n_2,\ldots,n_k})}(K_{n_1,n_2,\ldots,n_k}) = \emptyset$ then $K_{n_1,n_2,\ldots,n_k}$ is an ACLM graph as explained in the previous section. Thus we hereafter assume that $\mathcal{N}_{D(K_{n_1,n_2,\ldots,n_k})}(K_{n_1,n_2,\ldots,n_k}) \neq \emptyset$. Let p, q, r, s be any four vertices such that $E(K_{n_1,n_2,\ldots,n_k})$ contains $\{p, q\}$ and $\{r, s\}$ but neither $\{p, r\}$ nor $\{q, s\}$. Let G_1 be the graph obtained from $K_{n_1,n_2,\ldots,n_k}$ by removing $\{p, q\}$, G_2 be the graph obtained from G_1 by adding $\{p, r\}$, G_3 be the graph obtained from G_2 by removing $\{r, s\}$, and G be the graph obtained from G_3 by adding $\{q, s\}$. Then it is obvious that $G \in \mathcal{N}_{D(K_{n_1,n_2,\ldots,n_k})}$. We show in the following that $\lambda_2(G) \leq \lambda_2(K_{n_1,n_2,\ldots,n_k})$. By Interlace theorem [7], we have

$$\begin{aligned}
\lambda_2(G_1) \leq \lambda_2(K_{n_1,n_2,\ldots,n_k}) \leq \lambda_3(G_1) \\
\leq \lambda_3(K_{n_1,n_2,\ldots,n_k}) \leq \lambda_4(G_1) \leq \lambda_4(K_{n_1,n_2,\ldots,n_k}), \\
\lambda_3(G_1) \leq \lambda_3(G_2) \leq \lambda_4(G_1), \\
\lambda_3(G_3) \leq \lambda_3(G_2) \leq \lambda_4(G_3), \\
\lambda_2(G) \leq \lambda_3(G_3) \leq \lambda_3(G) \leq \lambda_4(G_3) \leq \lambda_4(G).
\end{aligned}$$

From these inequalities, we have $\lambda_2(G) \leq \lambda_4(K_{n_1,n_2,\ldots,n_k})$. Also, it follows from the assumption $n_k \geq 4$ that the multiplicity $n_k - 1$ of $\lambda_2(K_{n_1,n_2,\ldots,n_k})$ is at least three, which means that $\lambda_4(K_{n_1,n_2,\ldots,n_k}) = \lambda_2(K_{n_1,n_2,\ldots,n_k})$. Therefore, we conclude that $\lambda_2(G) \leq \lambda_2(K_{n_1,n_2,\ldots,n_k})$. □

We next give the second main theorem of this paper, which takes a different approach to show that any $K_{n_1,n_2,\ldots,n_k}$ with $n_{k-1} \geq 2$ and $n_k \geq 3$ is an ACLM graph.

Theorem 2 Any $K_{n_1,n_2,\ldots,n_k}$ is an ACLM graph in $\mathcal{G}_{D(K_{n_1,n_2,\ldots,n_k})}$. Furthermore, if $n_{k-1} \geq 2$ and $n_k \geq 3$, there exists at least one graph G in $\mathcal{N}_{D(K_{n_1,n_2,\ldots,n_k})}(K_{n_1,n_2,\ldots,n_k})$ such that

$$\lambda_2(G) \leq \lambda_2(K_{n_1,n_2,\ldots,n_k}) - 1 + \frac{2}{n_k} \leq \lambda_2(K_{n_1,n_2,\ldots,n_k}) - \frac{1}{3}. \quad (1)$$

Proof: Because it has already been shown by Corollary 1 that $K_{n_1,n_2,\ldots,n_k}$ is an ACM graph in $\mathcal{G}_{D(K_{n_1,n_2,\ldots,n_k})}$ if $n_{k-1} = 1$ or $n_{k-1} = n_k = 2$, we assume in the following that $n_{k-1} \geq 2$ and $n_k \geq 3$. Let p, q, r, s be any four vertices such that $E(K_{n_1,n_2,\ldots,n_k})$ contains $\{p, q\}$ and $\{r, s\}$ but neither $\{p, r\}$ nor $\{q, s\}$ (Existence of such four vertices is guaranteed by the assumption that $n_{k-1} \geq 2$ and $n_k \geq 3$). Then we easily see that

$$\pi(p) = \pi(r) \neq \pi(q) = \pi(s). \quad (2)$$

Let G be the graph obtained from $K_{n_1,n_2,\ldots,n_k}$ by applying a 2-switch to four vertices p, q, r and s as shown in Fig. 1. Then the Laplacian matrix of G is given by

$$L(G) = L(K_{n_1,n_2,\ldots,n_k}) - M$$

where $M = (m_{ij})$ is given by

$$m_{ij} = \begin{cases} -1, & \text{if } (i, j) \in \{(p, q), (q, p), (r, s), (s, r)\}, \\ 1, & \text{if } (i, j) \in \{(p, r), (r, p), (q, s), (s, q)\}, \\ 0, & \text{otherwise}. \end{cases}$$

Since $\lambda_2(G)$ is expressed as

$$\lambda_2(G) = \min_{\boldsymbol{v}^{\mathrm{T}}\mathbf{1}=0,\, \|\boldsymbol{v}\|=1} \boldsymbol{v}^{\mathrm{T}} L(G) \boldsymbol{v},$$

we can find an upper bound for $\lambda_2(G)$ by substituting any $\boldsymbol{v}$ such that $\boldsymbol{v}^{\mathrm{T}}\mathbf{1} = 0$ and $\|\boldsymbol{v}\| = 1$ into $\boldsymbol{v}^{\mathrm{T}} L(G) \boldsymbol{v}$. Let us now assume that $\boldsymbol{v}$ satisfies the condition that $v_i = 0$ for all $i \notin V_k$ in addition to $\boldsymbol{v}^{\mathrm{T}}\mathbf{1} = 0$ and $\|\boldsymbol{v}\| = 1$. Then, by Lemma 2, $\boldsymbol{v}$ is a unit eigenvector of $L(K_{n_1,n_2,\ldots,n_k})$ associated with $\lambda_2(K_{n_1,n_2,\ldots,n_k})$. We therefore have

$$\begin{aligned} \lambda_2(G) &\leq \boldsymbol{v}^{\mathrm{T}} L(G) \boldsymbol{v} \\ &= \lambda_2(K_{n_1,n_2,\ldots,n_k}) - \boldsymbol{v}^{\mathrm{T}} M \boldsymbol{v} \\ &= \lambda_2(K_{n_1,n_2,\ldots,n_k}) \\ &\quad + 2v_p v_q + 2v_r v_s - 2v_p v_r - 2v_q v_s \,. \end{aligned} \tag{3}$$

We first consider the case where $p \notin V_k$ (or $\pi(p) \neq k$) and $q \notin V_k$ (or $\pi(q) \neq k$). In this case, because $v_p = v_q = v_r = v_s = 0$, it follows from (3) that $\lambda_2(G) \leq \lambda_2(K_{n_1,n_2,\ldots,n_k})$. We next consider the case where either $p \in V_k$ (or $\pi(p) = k$) or $q \in V_k$ (or $\pi(q) = k$) holds. We assume without loss of generality that the former holds. In this case, because $v_q = v_s = 0$, it follows from (3) that $\lambda_2(G) \leq \lambda_2(K_{n_1,n_2,\ldots,n_k}) - 2v_p v_r$. In the following, we focus our attention on the minimum value of the second term $-2v_p v_r$ under the constraints on $\boldsymbol{v}$ mentioned above. This minimum value can be found by solving the mathematical programming problem:

$$\begin{array}{ll} \text{minimize} & -2v_p v_r \\ \text{subject to} & \sum_{i \in V_k} v_i = 0\,, \\ & \sum_{i \in V_k} v_i^2 = 1\,. \end{array} \tag{4}$$

Using the method of Lagrange multiplier, we obtain an optimal solution of (4) which is given by

$$v_i^* = \begin{cases} \frac{\mu_1}{2(1-\mu_2)}, & \text{if } i \in \{p, r\}, \\ -\frac{\mu_1}{2\mu_2}, & \text{if } i \in V_k \setminus \{p, r\} \end{cases} \tag{5}$$

where

$$\mu_1 = \sqrt{\frac{8(n_k - 2)}{n_k^3}} \quad \text{and} \quad \mu_2 = \frac{n_k - 2}{n_k}\,.$$

Substituting (5) to the objective function of (4), we have

$$-2v_p v_r = -1 + \frac{2}{n_k}\,.$$

Therefore, we have

$$\lambda_2(G) \leq \lambda_2(K_{n_1,n_2,\ldots,n_k}) - 1 + \frac{2}{n_k}$$

which completes the proof. □

In order to see how tight the upper bound given in (1) is, we consider a graph that can be obtained from $K_{1,1,2,4}$ shown in Fig. 2 by applying a 2-switch. Removing edges $\{3,5\}$ and $\{4,6\}$ and adding new edges $\{3,4\}$ and $\{5,6\}$, we have the graph shown in Fig. 3. The algebraic connectivity of the obtained graph is 3.2679492, while the right-hand side of (1) is given by

$$\lambda_2(K_{1,1,2,4}) - 1 + \frac{2}{4} = 3.5$$

which is slightly greater than the true value.

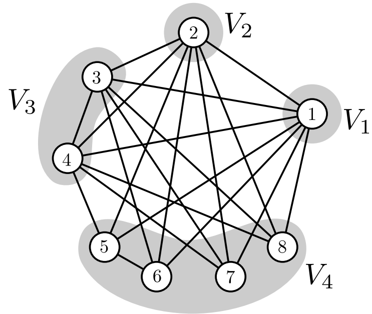

Figure 3: The graph obtained from $K_{1,1,2,4}$ shown in Fig. 2 by removing $\{3,5\}$ and $\{4,6\}$ and adding $\{3,4\}$ and $\{5,6\}$. The algebraic connectivity of this graph is 3.2679492.

4. Conclusion

We have proved that any complete multipartite graph $K_{n_1,n_2,\ldots,n_k}$ is an ACLM graph in the space of graphs with the same degree sequence. We have also proved that if $n_{k-1} \geq 2$ and $n_k \geq 3$ then there are four vertices in $K_{n_1,n_2,\ldots,n_k}$ such that the application of a 2-switch to the four vertices decreases the algebraic connectivity by at least $1/3$. One of the future problems is to prove that any complete multipartite graph is an ACM graph.

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