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Proposal of a Truly Decentralized Algorithm for Estimating Algebraic Connectivity of Multi-Agent Networks

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Abstract—Algebraic connectivity, the second smallest eigenvalue of the Laplacian matrix, of a network is an important quantity that represents how well the network is connected. In this paper, we propose a novel method for each agent in a network to estimate the algebraic connectivity of the network. The proposed method is truly decentralized because each agent updates the state value by using the information obtained from only agents in the neighborhood. The validity of the proposed method is confirmed by theoretical analysis and numerical experiments.

1. Introduction

One of the fundamental issues in mobile agent networks is how to keep the connectivity [1]. A promising approach to this problem is to make use of the algebraic connectivity, which is defined as the second smallest eigenvalue of the Laplacian matrix [2]. The algebraic connectivity is an important measure that represents how well the network is connected. In particular, it takes a positive value if and only if the network is connected. However, it is not so easy for agents to compute or estimate the algebraic connectivity because each agent in general cannot communicate with all agents in the network.

Recently, Yang *et al.* [3] proposed a continuous-time algorithm for agents to estimate the algebraic connectivity of the network. They also proved analytically that the value of the algebraic connectivity estimated by each agent converges to the true one for almost all initial conditions. A few years later, Fukami and Takahashi [4] proposed a modified algorithm, which is more suitable for hardware implementation, and proved that the algebraic connectivity can be estimated by their algorithm too. However, these two algorithms are not decentralized in a strict sense, because they are based on a strong assumption that each agent can compute the average of the state values of all agents instantaneously. Yang *et al.* claim that the computation of the average can be done quickly by using some average consensus algorithm [5] with a very small time constant, but it is not clear how to initialize the consensus algorithm.

In this paper, we propose a novel continuous-time algorithm for the estimation of the algebraic connectivity, and show the validity from both theoretical and experimental points of view. The proposed algorithm is a combination of the one given by Yang *et al.* [3] and a dynamic average consensus algorithm. Here, by dynamic average consensus, we mean that the state value of each agent tracks the average of multiple time-varying reference signals [6, 7]. Because each agent updates the state value by using the information obtained from only agents in the neighborhood, the proposed algorithm is truly distributed.

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2. Conventional Algorithm

Let us consider a network of n agents labeled from 1 to n that communicate with each other. Throughout this paper, we assume that the communication topology is static and symmetric, that is, if agent i can communicate with agent j then agent j can communicate with agent i. Under these assumptions, the communication between agents is expressed by an undirected graph $G = (V, E)$ where $V = \{1, 2, \ldots, n\}$ is the set of vertices representing agents and E is the set of undirected edges such that $\{i, j\} \in E$ if and only if agents i and j can communicate with each other. The set of all vertices adjacent to vertex i is called the neighborhood of vertex i and denoted by $\mathcal{N}_i$. The adjacency matrix is denoted by $\boldsymbol{A} = (a_{ij}) \in \{1, 0\}^{n\times n}$ where $a_{ij} = 1$ if and only if $\{i, j\} \in E$. The degree matrix is denoted by $\boldsymbol{D} = \mathrm{diag}(d_1, d_2, \ldots, d_n) \in \mathbb{Z}_+^{n\times n}$ where $\mathbb{Z}_+$ is the set of nonnegative integers and $d_i = |\mathcal{N}_i|$ for $i = 1, 2, \ldots, n$. The Laplacian matrix is defined by $\boldsymbol{L} = \boldsymbol{D} - \boldsymbol{A}$. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be n eigenvalues of $\boldsymbol{L}$. Because $\boldsymbol{L}$ is a real symmetric matrix, its eigenvalues are real. Hence we assume without loss of generality that $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. Also, it is well known that λ_1 is always zero and λ_2 is nonzero if and only if G is connected. Let $\boldsymbol{p}_i$ be a normalized eigenvector associated with λ_i for $i = 1, 2, \ldots, n$. Because $\boldsymbol{L}\boldsymbol{1} = \boldsymbol{0}$, where $\boldsymbol{1}$ and $\boldsymbol{0}$ are vectors of all 1 and all 0, respectively, $\boldsymbol{p}_1$ is either $\frac{1}{\sqrt{n}}\boldsymbol{1}$ or $-\frac{1}{\sqrt{n}}\boldsymbol{1}$.

The second smallest eigenvalue λ_2 of $\boldsymbol{L}$, which is called the algebraic connectivity, reflects how well the graph is connected. Yang *et al.* [3] proposed a continuous-time algorithm for agents to estimate the algebraic connectivity of the graph representing the communication between agents. Their algorithm is expressed by the following system of differential equations:

$$\dot{x}_i = -k_1\left(\frac{1}{n}\sum_{j=1}^{n} x_j\right) - k_2\sum_{j\in\mathcal{N}_i}(x_i - x_j) - k_3\left(\frac{1}{n}\sum_{j=1}^{n} x_j^2 - 1\right)x_i, \quad i = 1,2,\ldots,n \quad (1)$$

where $x_i(t)$ is the state value of agent i at time t and k_i $(i = 1, 2, 3)$ are positive constants. By introducing $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}$, (1) can be rewritten in a vector form as follows:

$$\dot{\boldsymbol{x}} = -k_1\left(\frac{1}{n}\mathbf{1}^{\mathrm{T}}\boldsymbol{x}\right)\mathbf{1} - k_2\boldsymbol{L}\boldsymbol{x} - k_3\left(\frac{1}{n}\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x} - 1\right)\boldsymbol{x}. \quad (2)$$

Yang *et al.* analyzed the dynamical behavior of (2) and derived the following theorem.

Theorem 1 (Yang et al. [3]) Suppose that the network is connected and the initial value $\boldsymbol{x}(0)$ satisfies $\boldsymbol{p}_2^{\mathrm{T}}\boldsymbol{x}(0) \neq 0$. Then any solution $\boldsymbol{x}(t)$ of (2) converges to either $\mu\boldsymbol{p}_2$ or $-\mu\boldsymbol{p}_2$ where μ is a positive constant given by

$$\mu = \sqrt{\frac{n(k_3 - k_2\lambda_2)}{k_3}}$$

if and only if $k_1 > \lambda_2 k_2$ and $k_3 > \lambda_2 k_2$.

From Theorem 1 we can easily see that

$$\lim_{t\to\infty}\frac{\sum_{j\in\mathcal{N}_i}(x_i(t) - x_j(t))}{x_i(t)} = \lim_{t\to\infty}\frac{(\boldsymbol{L}\boldsymbol{x}(t))_i}{x_i(t)} = \frac{(\boldsymbol{L}(\mu\boldsymbol{p}_2))_i}{\mu p_{2i}} = \lambda_2 \quad (3)$$

which means that the algebraic connectivity λ_2 can be obtained from the solution $\boldsymbol{x}(t)$ of (1). However, it is assumed in (1) that the values of $\sum_{j=1}^{n} x_j(t)$ and $\sum_{j=1}^{n} x_i(t)^2$ can be computed instantaneously. This assumption cannot be met in general because each agent does not necessarily communicate with all other agents. Yang *et al.* claim that the values of $\sum_{j=1}^{n} x_j(t)$ and $\sum_{j=1}^{n} x_i(t)^2$ can be computed almost instantaneously by using some consensus algorithm, such as the one proposed by Olfati-Saber and Murray [5], with a much smaller time constant than the main algorithm (1). However, there still remain some problems to be solved, such as how to set the initial value of the consensus algorithm.

3. Proposed Algorithm

3.1. Dynamic Average Consensus Algorithm

In order to develop a truly decentralized algorithm for the estimation of the algebraic connectivity of the network, a dynamic average consensus algorithm is needed. We propose to use the continuous-time algorithm given by

$$x_i(t) = \hat{x}_i(t) + r_i(t), \quad i = 1,2,\ldots,n, \quad (4)$$

$$\dot{\hat{x}}_i = \alpha\sum_{j\in\mathcal{N}_i}(x_j - x_i), \quad i = 1,2,\ldots,n, \quad (5)$$

$$\hat{x}_i(0) = 0, \quad i = 1,2,\ldots,n, \quad (6)$$

where $r_i(t)$ is the time-varying input signal given to agent i, and $(x_i(t), \hat{x}_i(t))$ is the state vector of agent i at time t. This algorithm is based on the one proposed by Chen *et al.* [6], but different from it because the signum function is not used in (5).

Theorem 2 Suppose that the network is connected and there exist positive constants C and b such that

$$\forall t \geq 0, \quad \|\dot{\boldsymbol{r}}(t)\| \leq Ce^{-bt} \quad (7)$$

where $\boldsymbol{r}(t) = (r_1(t), r_2(t), \ldots, r_n(t))^{\mathrm{T}}$. Then the algorithm given by (4)–(6) satisfies

$$\lim_{t\to\infty}\left(x_i(t) - \frac{1}{n}\sum_{j=1}^{n} r_j(t)\right) = 0, \quad i = 1,2,\ldots,n. \quad (8)$$

Proof: Eliminating $\hat{x}_1(t), \hat{x}_2(t), \ldots, \hat{x}_n(t)$ from (4)–(6), we have

$$\dot{\boldsymbol{x}} = -\alpha\boldsymbol{L}\boldsymbol{x} + \dot{\boldsymbol{r}}, \quad (9)$$

$$\boldsymbol{x}(0) = \boldsymbol{r}(0), \quad (10)$$

where $\boldsymbol{x}(t) = (x_1(t), x_2(t), \ldots, x_n(t))^{\mathrm{T}}$ and $\boldsymbol{L}$ is the Laplacian matrix of the graph. Note that $\boldsymbol{L}$ is decomposed as $\boldsymbol{L} = \boldsymbol{P}\boldsymbol{\Lambda}\boldsymbol{P}^{\mathrm{T}}$ where $\boldsymbol{P} = (\boldsymbol{p}_1, \boldsymbol{p}_2, \ldots, \boldsymbol{p}_n)$ and $\boldsymbol{\Lambda} = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ with $\lambda_1 = 0$. Note also that $\lambda_2, \lambda_3, \ldots, \lambda_n$ are positive because the network is assumed to be connected. Multiplying $\boldsymbol{P}^{\mathrm{T}}$ from the left to both sides of (9) and (10), and putting $\boldsymbol{P}^{\mathrm{T}}\boldsymbol{x}(t) = \boldsymbol{y}(t) = (y_1(t), y_2(t), \ldots, y_n(t))^{\mathrm{T}}$, we have

$$\dot{\boldsymbol{y}} = -\alpha\boldsymbol{\Lambda}\boldsymbol{y} + \boldsymbol{P}^{\mathrm{T}}\dot{\boldsymbol{r}}, \quad (11)$$

$$\boldsymbol{y}(0) = \boldsymbol{P}^{\mathrm{T}}\boldsymbol{r}(0). \quad (12)$$

The solution of this differential equation is given by

$$y_1(t) = \boldsymbol{p}_1^{\mathrm{T}}\boldsymbol{r}(t) \quad (13)$$

and

$$y_i(t) = e^{-\alpha\lambda_i t}\int_0^t e^{\alpha\lambda_i s}\boldsymbol{p}_i^{\mathrm{T}}\dot{\boldsymbol{r}}(s)\mathrm{d}s + y_i(0)e^{-\alpha\lambda_i t}, \quad i = 2,3,\ldots,n. \quad (14)$$

From (14), we have

$$\begin{aligned}
|y_i(t)| &\leq e^{-\alpha\lambda_i t}\int_0^t e^{\alpha\lambda_i s}\|\boldsymbol{p}_i\|\|\dot{\boldsymbol{r}}(s)\|\mathrm{d}s + y_i(0)e^{-\alpha\lambda_i t} \\
&\leq e^{-\alpha\lambda_i t}\int_0^t e^{\alpha\lambda_i s}Ce^{-bs}\mathrm{d}s + y_i(0)e^{-\alpha\lambda_i t} \\
&= Ce^{-\alpha\lambda_i t}\int_0^t e^{(\alpha\lambda_i - b)s}\mathrm{d}s + y_i(0)e^{-\alpha\lambda_i t} \\
&= y_i(0)e^{-\alpha\lambda_i t} + \begin{cases} Cte^{-\alpha\lambda_i t}, & \text{if } \alpha\lambda_i = b, \\ \frac{C}{\alpha\lambda_i - b}(e^{-bt} - e^{-\alpha\lambda_i t}), & \text{if } \alpha\lambda_i \neq b. \end{cases}
\end{aligned}$$

Because $\lambda_2, \lambda_3, \ldots, \lambda_n$ are positive, we have

$$\lim_{t\to\infty} y_i(t) = 0, \quad i = 2, 3, \ldots, n\,. \tag{15}$$

It follows from (13) and (15) that the solution of (11) and (12) satisfies

$$\lim_{t\to\infty} \left(\boldsymbol{y}(t) - (\boldsymbol{p}_1^{\mathrm{T}}\boldsymbol{r}(t), 0, 0, \ldots, 0)^{\mathrm{T}}\right) = \boldsymbol{0}\,.$$

Multiplying $\boldsymbol{P}$ from the left to both sides of this equation, we have

$$\lim_{t\to\infty}(\boldsymbol{x}(t) - \boldsymbol{p}_1\boldsymbol{p}_1^{\mathrm{T}}\boldsymbol{r}(t)) = \boldsymbol{0}$$

which is equivalent to (8). □

3.2. Algebraic Connectivity Estimation Algorithm

We now propose a new continuous-time algorithm which is obtained from (1) by replacing the computations of $\frac{1}{n}\sum_{j=1}^{n} x_j(t)$ and $\frac{1}{n}\sum_{j=1}^{n} x_j(t)^2$ with the dynamic average consensus algorithm given by (4)–(6). It is expressed by the following system of differential equations:

$$\begin{aligned}
\dot{x}_i(t) &= -k_1 y_i(t) - k_2 \sum_{j\in\mathcal{N}_i}(x_j(t) - x_i(t)) \\
&\quad -k_3(z_i(t) - 1)x_i(t), \quad i = 1, 2, \ldots, n && (16)\\
y_i(t) &= \hat{y}_i(t) + x_i(t), \quad i = 1, 2, \ldots, n && (17)\\
\dot{\hat{y}}_i(t) &= \alpha \sum_{j\in\mathcal{N}_i}(y_j(t) - y_i(t)), \quad i = 1, 2, \ldots, n && (18)\\
\hat{y}_i(0) &= 0, \quad i = 1, 2, \ldots, n && (19)\\
z_i(t) &= \hat{z}_i(t) + x_i^2(t), \quad i = 1, 2, \ldots, n && (20)\\
\dot{\hat{z}}_i(t) &= \alpha \sum_{j\in\mathcal{N}_i}(z_j(t) - z_i(t)), \quad i = 1, 2, \ldots, n && (21)\\
\hat{z}_i(0) &= 0, \quad i = 1, 2, \ldots, n && (22)
\end{aligned}$$

where $(x_i(t), y_i(t), \hat{y}_i(t), z_i(t), \hat{z}_i(t))^{\mathrm{T}}$ is the state vector of agent i at time t. The first element $x_i(t)$ represents the estimated value of the algebraic connectivity. The second element $y_i(t)$ and the fourth element $z_i(t)$ represent the estimated values of $\frac{1}{n}\sum_{j=1}^{n} x_j(t)$ and $\frac{1}{n}\sum_{j=1}^{n} x_j(t)^2$, respectively. Because each agent only needs information of itself and its neighbors in order to update the state vector, the proposed algorithm is truly decentralized.

If the network is connected and $\boldsymbol{x}(t)$ satisfies

$$\forall t \geq 0, \quad \|\dot{\boldsymbol{x}}(t)\| \leq Ce^{-bt} \tag{23}$$

for some positive constants C and b, then it follows from Theorem 2 that $y_i(t) - \frac{1}{n}\sum_{j=1}^{n} x_j(t)$ converges to 0 for all i and $z_i(t) - \frac{1}{n}\sum_{j=1}^{n} x_j(t)^2$ converges to 0 for all i. In this case, it is expected that $\boldsymbol{x}(t)$ converges to an eigenvector of $\boldsymbol{L}$ associated with λ_2 and that the algebraic connectivity can be estimated by (3). However, it is not clear whether (23) is true or false because the behavior of $\boldsymbol{x}(t)$ depends on $y_1(t), y_2(t), \ldots, y_n(t)$ and $z_1(t), z_2(t), \ldots, z_n(t)$. Theoretical analysis of the convergence property of the proposed algorithm is a future problem.

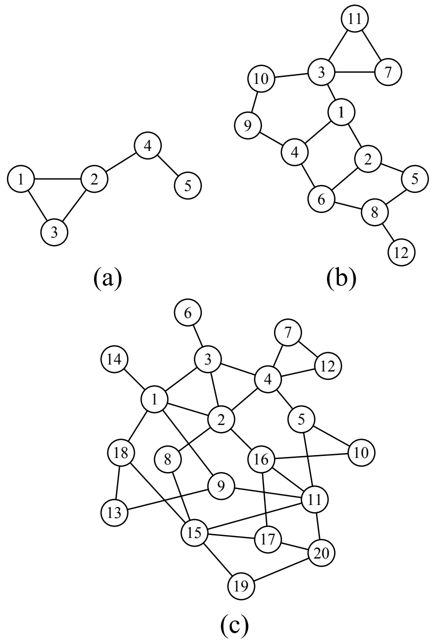

Figure 1: Networks used in numerical experiments. Algebraic connectivities are: (a) 0.5188057, (b) 0.1794688, and (c) 0.4165522.

4. Numerical Experiments

In order to examine the validity of the proposed method, we conducted some numerical experiments. To be more specific, for each of the three networks shown in Fig.1, we found the solution $\boldsymbol{x}(t)$ of the system of differential equations described by (16)–(22) numerically by using Euler's method with a step size of 0.005, and then checked whether $\boldsymbol{x}(t)$ converged to an eigenvector of $\boldsymbol{L}$ associated with λ_2 and whether λ_2 was correctly estimated by (3). The values of positive constants k_1, k_2, k_3 and α were set to 1, and the initial value $\boldsymbol{x}(0)$ was determined randomly.

Results are shown in Figs.2–5. Fig.2 shows the waveforms of $|\cos\theta(t)| = |\boldsymbol{x}(t)^{\mathrm{T}}\boldsymbol{p}_2|/\|\boldsymbol{x}(t)\|$ and $\|\boldsymbol{x}(t)\|$ for the network shown in Fig.1(a). Because $|\cos\theta(t)|$ converges to 1 and $\|\boldsymbol{x}(t)\|$ converges to a positive constant, we can conclude that $\boldsymbol{x}(t)$ converges to an eigenvector of $\boldsymbol{L}$ associated with λ_2. Fig.3 shows the waveforms of the values of the algebraic connectivity estimated by the five agents in the same network. We see there that all of the five estimated values converge to the true value. From these observations, we can conclude that the proposed algorithm works properly for the network shown in Fig.1(a). We also see from Figs.4 and 5 that the proposed algorithm works for other two networks in Fig.1.

5. Conclusion

We have proposed a new continuous-time algorithm for the estimation of the algebraic connectivity of multi-agent

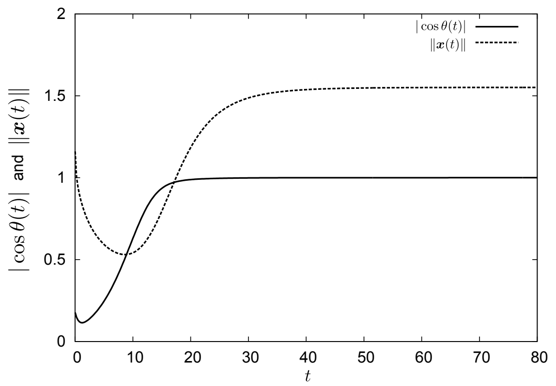

Figure 2: Waveforms of $|\cos\theta(t)| = |\boldsymbol{x}(t)^{\mathrm{T}}\boldsymbol{p}_2|/\|\boldsymbol{x}(t)\|$ and $\|\boldsymbol{x}(t)\|$ for the network shown in Fig.1(a).

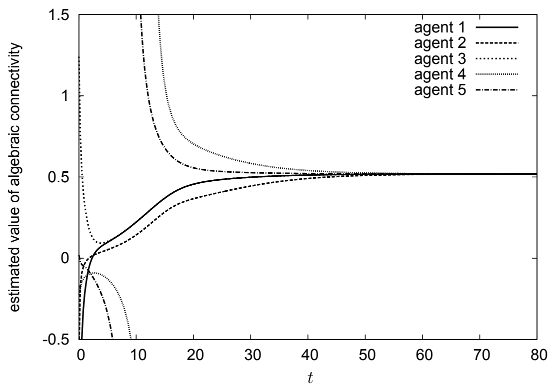

Figure 3: Waveforms of $\sum_{j\in\mathcal{N}_i}(x_i(t) - x_j(t))/x_i(t)$ $(i = 1, 2, 3, 4, 5)$ for the network shown in Fig.1(a).

networks. The validity of the the proposed algorithm has been confirmed by numerical experiments with three networks consisting of 5, 12 and 20 agents. However, it remains an open question whether the solution of the system of differential equations described by (16)–(22) always converges to an eigenvector of the Laplacian matrix associated with the algebraic connectivity. This question will be answered in future studies.

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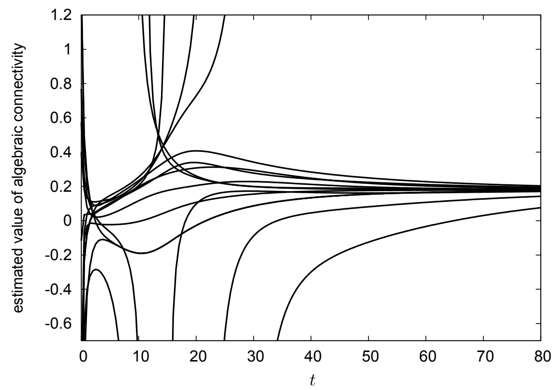

Figure 4: Waveforms of $\sum_{j\in\mathcal{N}_i}(x_i(t) - x_j(t))/x_i(t)$ $(i = 1, 2, \ldots, 12)$ for the network shown in Fig.1(b).

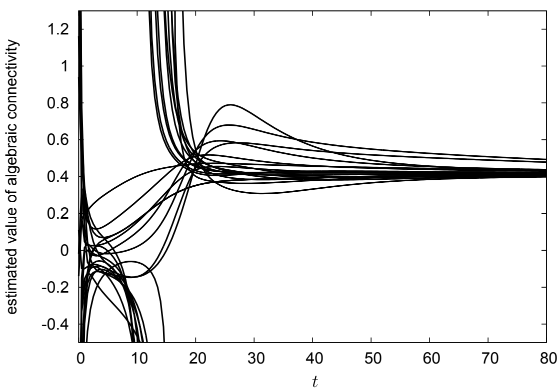

Figure 5: Waveforms of $\sum_{j\in\mathcal{N}_i}(x_i(t) - x_j(t))/x_i(t)$ $(i = 1, 2, \ldots, 20)$ for the network shown in Fig.1(c).

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