



Implementation of implicit Runge-Kutta methods into circuit simulators for nonlinear circuits

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Abstract—This paper describes a method of implementing implicit Runge-Kutta(IRK) formulas into circuit simulators for nonlinear circuits as numerical integration. Equivalent circuits at discrete time of IRK formulas for linear and nonlinear elements are proposed. In IRK methods, the size of equivalent circuit becomes larger than the numerical integration in conventional circuit simulators, because the equivalent circuit at an intermediate time between past and present time are needed in addition to the equivalent circuit at present time. This problem is solved by using larger time step for numerical integration compared to conventional methods to save calculation time, since RadauIIA and LobattoIIIA of IRK formulas have high orders and are A-stable. The IRK and conventional methods are compared in terms of accuracy and computational costs by using an example circuit.

1. Introduction

In conventional time-domain circuit simulators, multi-step backward-differential-formulas such as Gear's methods and trapezoidal algorithm are used to solve ordinary differential equations [1]-[3]. The order of trapezoidal formula is 2nd, and that of typical algorithms in Gear's methods are 1st or 2nd at most, since Gear's methods over 3rd order are not A-stable. RadauIIA(s=2) and LobattoIIIA(s=3) of IRK have 3rd order and 4th order, respectively[4]. They are A-stable and the stability region of IRK formulas includes the whole left half-complex plane. General formula for RadauIIA and LobattoIIIA of IRK formulas were introduced as numerical integration for circuit simulation in the reference [5]. In this paper, equivalent circuits and equations for the simulation for linear and nonlinear elements to implement IRK methods into a circuit simulator for general purposes are proposed. Then, it is shown that the size of an equivalent circuit becomes twice as large as circuits using conventional methods, as the equivalent circuit includes a circuit at a time between past and present in addition to the present circuit. Therefore, we compared the IRK and conventional methods in terms of accuracy and computational costs using an example circuit. Fortunately, we can use larger time steps for numerical integration than the ones used in conventional methods to save calculation time, since the order of IRK algorithms is high.

2. General formula for Runge-Kutta methods

For ordinary differential equation

$$y' = f(t, y),$$

Runge-Kutta methods are given by

$$\begin{cases} Y_i = y_n + h \sum_{j=1}^s a_{ij} f(t_n + c_j h, Y_j) & (i = 1, 2, \dots, s) \\ y_{n+1} = y_n + h \sum_{i=1}^s b_i f(t_n + c_i h, Y_i) \end{cases} \quad (1)$$

where each coefficient of (1) is given by Butcher arrangement shown in Table 1. In RadauIIA and LobattoIIIA, the upper triangle elements a_{ij} ($i \leq j$) are not zero and then Runge-Kutta methods are *implicit* numerical integrations. Parameters c_i denote the intermediate time. RadauIIA(s=2) and LobattoIIIA(s=3), which are referred to as Radau-s2 and Lobatto-s3, respectively, have one intermediate time, because $c_1 = \frac{1}{3}$, $c_2 = 1$ for Radau-s2 and $c_1 = 0$, $c_2 = \frac{1}{2}$, $c_3 = 1$ for Lobatto-s3. Assuming that $m_{ij} = a_{ij} m_{10} = m_{20} = 0$ in Radau-s2 and $m_{ij} = a_{(i-1)(j-1)}$ in Lobatto-s3, the two IRK methods can be written as follows [5]:

$$\begin{cases} Y(t_a) - y(t_n) - hm_{10}f(t_n) - hm_{11}f(t_a) - hm_{12}f(t_{n+1}) = 0 \\ y(t_{n+1}) - y(t_n) - hm_{20}f(t_n) - hm_{21}f(t_a) - hm_{22}f(t_{n+1}) = 0 \end{cases} \quad (2)$$

where t_n , t_a and t_{n+1} indicate a past time, an intermediate time and a present time.

Table 1: Butcher arrangement

c_1	a_{11}	a_{12}	\dots	a_{1s}
c_2	a_{21}	a_{22}	\dots	a_{2s}
\vdots	\vdots	\vdots	\ddots	\vdots
c_s	a_{s1}	a_{s2}	\dots	a_{ss}
	b_1	b_2	\dots	b_s

3. Equivalent circuit for a linear element

3.1. Linear capacitor

Applying (2) to ordinary differential equation $v'_c(t) = i(t)/C$ for a linear capacitor, we can get equation (3). Then, the equivalent circuit is shown in Fig. 1 and the equation

for the simulation is given by

$$\begin{bmatrix} B & -B & A & -A \\ -B & B & -A & A \\ E & -E & D & -D \\ -E & E & -D & D \end{bmatrix} \begin{bmatrix} v_{1,n+1} \\ v_{2,n+1} \\ v_{1,a} \\ v_{2,a} \end{bmatrix} = \begin{bmatrix} -P \\ P \\ -F \\ F \end{bmatrix} \quad (4)$$

where $v_{1,a}$ and $v_{2,a}$ are the voltages at an intermediate time t_a while $v_{1,n+1}$ and $v_{2,n+1}$ are the voltages at a present time t_{n+1} .

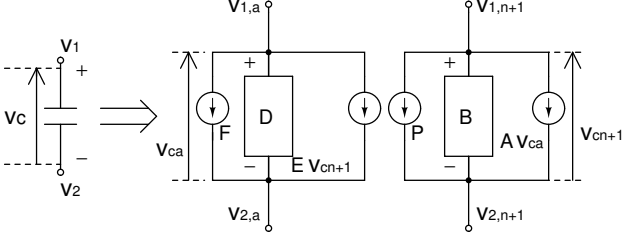


Figure 1: Equivalent circuit of linear capacitor for Radau-s2 and Lobatto-s3

Parameters A and E denote the coefficients of voltage controlled current source (VCCS). The VCCS with coefficient A is controlled by v_{ca} and E is controlled by v_{cn+1} . They are calculated by

$$A = \frac{-m_{21}}{\frac{h}{C}(m_{11}m_{22} - m_{12}m_{21})}, \quad E = \frac{-m_{12}}{\frac{h}{C}(m_{11}m_{22} - m_{12}m_{21})}.$$

Parameters B and D show admittances that are coefficients of the voltage at each moment. They are calculated by

$$B = \frac{m_{11}}{\frac{h}{C}(m_{11}m_{22} - m_{12}m_{21})}, \quad D = \frac{m_{22}}{\frac{h}{C}(m_{11}m_{22} - m_{12}m_{21})}.$$

Parameters P and F are independent current sources that can be computed from the past voltage and current values.

$$P = \frac{(-m_{11} + m_{21})v_{cn} + \frac{h}{C}(m_{10}m_{21} - m_{11}m_{20})i_{cn}}{\frac{h}{C}(m_{11}m_{22} - m_{12}m_{21})}$$

$$F = \frac{(m_{12} - m_{22})v_{cn} - \frac{h}{C}(m_{10}m_{22} - m_{12}m_{20})i_{cn}}{\frac{h}{C}(m_{11}m_{22} - m_{12}m_{21})}$$

4. Equivalent circuit for diode

4.1. Diode model

A model of a diode is shown in Fig. 2. It consists of a contact resistance R_s , a nonlinear conductance $g(v)$ and a nonlinear capacitor $q(v)$. $g(v)$ and $q(v)$ are connected parallelly.

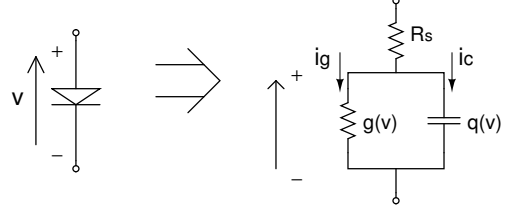


Figure 2: Diode model

4.2. Linearization and numerical integration for nonlinear capacitor

We assume that the relationship of the current i_c and the diode voltage v are given by

$$i_c = \frac{dq(v)}{dt} \quad (5)$$

$$q(v) = T_t I_s \left(\exp\left(\frac{v}{NV_t}\right) - 1 \right)$$

where T_t , I_s and N each shows a run time, a saturation current and an emission coefficient. Applying (2) to ordinary differential equation (5) and linearizing by Newton method, we reach equation (6), where (k) represents the number of Newton iteration. $C_a^{(k)}$, $C_{n+1}^{(k)}$, C_n and α in (6) are given by

$$C_a^{(k)} = \left. \frac{\partial q(v)}{\partial v} \right|_{v=v_a^{(k)}} \quad C_{n+1}^{(k)} = \left. \frac{\partial q(v)}{\partial v} \right|_{v=v_{n+1}^{(k)}}$$

$$C_n = \left. \frac{\partial q(v)}{\partial v} \right|_{v=v_n} \quad \alpha = (m_{11}m_{22} - m_{12}m_{21}).$$

Equation (6) is rewritten by followings:

$$\begin{cases} i_a = \frac{(m_{12} - m_{22})v_n + m_{22}v_a - m_{12}v_{n+1} - \frac{h}{C}(m_{10}m_{22} - m_{12}m_{20})i_n}{\frac{h}{C}(m_{11}m_{22} - m_{12}m_{21})} \\ i_{n+1} = \frac{(-m_{11} + m_{21})v_n - m_{21}v_a + m_{11}v_{n+1} + \frac{h}{C}(m_{10}m_{21} - m_{11}m_{20})i_n}{\frac{h}{C}(m_{11}m_{22} - m_{12}m_{21})} \end{cases} \quad (3)$$

$$\begin{cases} i_{ca}^{(k+1)} = \frac{m_{22}}{h\alpha} \left\{ q(v_a^{(k)}) + C_a^{(k)} (v_a^{(k+1)} - v_a^{(k)}) \right\} - \frac{m_{12}}{h\alpha} \left\{ q(v_{n+1}^{(k)}) + C_{n+1}^{(k)} (v_{n+1}^{(k+1)} - v_{n+1}^{(k)}) \right\} \\ \quad + \frac{1}{h\alpha} \left\{ (m_{12} - m_{22})q(v_n) - h(m_{10}m_{22} - m_{12}m_{20})C_n \right\} \\ i_{cn+1}^{(k+1)} = \frac{m_{11}}{h\alpha} \left\{ q(v_{n+1}^{(k)}) + C_{n+1}^{(k)} (v_{n+1}^{(k+1)} - v_{n+1}^{(k)}) \right\} - \frac{m_{21}}{h\alpha} \left\{ q(v_a^{(k)}) + C_a^{(k)} (v_a^{(k+1)} - v_a^{(k)}) \right\} \\ \quad + \frac{1}{h\alpha} \left\{ (m_{21} - m_{11})q(v_n) + h(m_{10}m_{21} - m_{11}m_{20})C_n \right\} \end{cases} \quad (6)$$

$$\begin{cases} i_{ca}^{(k+1)} = G_{ca}^{(k)} v_a^{(k+1)} + g_{cn+1}^{(k)} v_{n+1}^{(k+1)} + I_{ca}^{(k)} \\ i_{cn+1}^{(k+1)} = G_{cn+1}^{(k)} v_{n+1}^{(k+1)} + g_{ca}^{(k)} v_a^{(k+1)} + I_{cn+1}^{(k)} \end{cases} \quad (7)$$

where $G_{ca}^{(k)}$ and $G_{cn+1}^{(k)}$ are conductances and decided by

$$G_{ca}^{(k)} = \frac{m_{22}}{h\alpha} C_a^{(k)} \quad G_{cn+1}^{(k)} = \frac{m_{11}}{h\alpha} C_{n+1}^{(k)}$$

$g_{cn+1}^{(k)}$ and $g_{ca}^{(k)}$ in (7) show the coefficients of VCCS. The VCCS with coefficient $g_{cn+1}^{(k)}$ is connected to the equivalent circuit at t_a and controlled by voltage at t_{n+1} . $g_{ca}^{(k)}$ is connected to the equivalent circuit at t_{n+1} and controlled by voltage at t_a . They are determined by

$$g_{cn+1}^{(k)} = -\frac{m_{12}}{h\alpha} C_{n+1}^{(k)} \quad g_{ca}^{(k)} = -\frac{m_{21}}{h\alpha} C_a^{(k)}$$

$I_{ca}^{(k)}$ and $I_{cn+1}^{(k)}$ in (7) are given by

$$\begin{aligned} I_{ca}^{(k)} &= I_{c1}^{(k)} + i_{c1}^{(k)} + I_{c1n} \\ I_{cn+1}^{(k)} &= I_{c2}^{(k)} + i_{c2}^{(k)} + I_{c2n} \end{aligned}$$

where each term is calculated by

$$\begin{aligned} I_{c1}^{(k)} &= m_{22}/(h\alpha) \{q(v_a^{(k)}) + T_t I_s - C_a^{(k)} v_a^{(k)}\} \\ I_{c2}^{(k)} &= m_{11}/(h\alpha) \{q(v_{n+1}^{(k)}) + T_t I_s - C_{n+1}^{(k)} v_{n+1}^{(k)}\} \\ i_{c1}^{(k)} &= m_{12}/(h\alpha) \{C_{n+1}^{(k)} v_{n+1}^{(k)} - q(v_{n+1}^{(k)}) - T_t I_s\} \\ i_{c2}^{(k)} &= m_{21}/(h\alpha) \{C_a^{(k)} v_a^{(k)} - q(v_a^{(k)}) - T_t I_s\} \\ I_{c1n} &= T_t I_s (m_{12} - m_{22})/h\alpha \\ &+ \frac{T_t I_s}{h\alpha} [(m_{12} - m_{22}) - h(m_{10}m_{22} - m_{12}m_{20})] q(v_n) + T_t I_s - 1 \\ I_{c2n} &= T_t I_s (m_{21} - m_{11})/h\alpha \\ &+ \frac{T_t I_s}{h\alpha} [(m_{21} - m_{11}) + h(m_{10}m_{21} - m_{11}m_{20})] q(v_n) + T_t I_s - 1 \end{aligned}$$

4.3. Linearization for nonlinear conductance

Nonlinear conductance is linearized by applying Newton method at both t_a and t_{n+1} , as Radau-s2 and Lobatto-s3 have one intermediate point. Then, the equivalent circuit for nonlinear conductance consists of conductances $G_{ga}^{(k)}$ and $G_{gn+1}^{(k)}$ and independent current sources $I_{ga}^{(k)}$ and $I_{gn+1}^{(k)}$ which are parallel with each other, where

$$\begin{aligned} G_{ga}^{(k)} &= \left. \frac{\partial g(v)}{\partial v} \right|_{v=v_a^{(k)}} & G_{gn+1}^{(k)} &= \left. \frac{\partial g(v)}{\partial v} \right|_{v=v_{n+1}^{(k)}} \\ I_{ga}^{(k)} &= g(v_a^{(k)}) - G_{ga}^{(k)} v_a^{(k)} & I_{gn+1}^{(k)} &= g(v_{n+1}^{(k)}) - G_{gn+1}^{(k)} v_{n+1}^{(k)} \end{aligned}$$

4.4. Equivalent circuit for diode

An equivalent circuit at $(k+1)$ Newton iteration and $(n+1)$ discrete time for a diode is shown in Fig. 3, where

$$\begin{aligned} G_{d1}^{(k)} &= G_{ga}^{(k)} + G_{ca}^{(k)} & G_{d2}^{(k)} &= G_{gn+1}^{(k)} + G_{cn+1}^{(k)} \\ I_{d1}^{(k)} &= I_{ga}^{(k)} + I_{ca}^{(k)} & I_{d2}^{(k)} &= I_{gn+1}^{(k)} + I_{cn+1}^{(k)} \end{aligned}$$

Then, the equation for the simulation is given by equation (8).

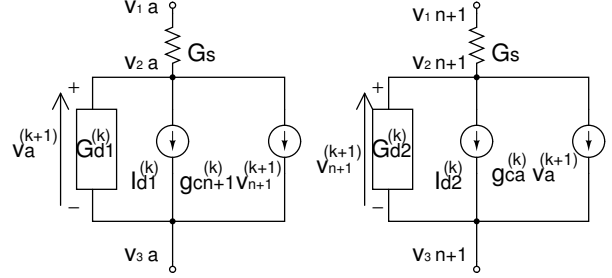


Figure 3: Equivalent circuit for diode

5. Comparison of the error with the conventional method

The estimated error of the trapezoidal, Radau-s2 and Lobatto-s3 for an RC circuit shown in Fig. 4, ε is calculated by

$$\varepsilon = \left| \frac{v_c - v}{v_c} \right| \times 100[\%]$$

where v_c is an exact solution at $t = 1$ [s] calculated from

$$v_c(t) = E \left(1 - \exp \frac{t}{RC} \right)$$

and v is a numerical solution of each numerical integration method. An exact solution is used as the past value to calculate the present numerical solution. The results are shown in Fig. 5, where horizontal axis shows the time step and vertical axis shows the error. According to the graph, larger time steps can be used in IRK methods in order to obtain the same accuracy as the trapezoidal rule. For example, Radau-s2 can use 6 times the numerical integration time step as large as the trapezoidal rule and Lobatto-s3 can use 77 times as large time step as the trapezoidal with 1.0×10^{-4} [%] error.

$$\begin{bmatrix} G_s & -G_s & & & & & \\ -G_s & G_s + G_{d2}^{(k)} & -G_{d2}^{(k)} & & & & \\ & -G_{d2}^{(k)} & G_{d2}^{(k)} & & & & \\ & & & G_s & -G_s & & \\ & & & -G_s & G_s + G_{d1}^{(k)} & -G_{d1}^{(k)} & \\ & & & & -G_{d1}^{(k)} & G_{d1}^{(k)} & \\ & & & & & & G_{cn+1}^{(k)} & -g_{cn+1}^{(k)} & -G_{cn+1}^{(k)} & & \\ & & & & & & -g_{cn+1}^{(k)} & g_{cn+1}^{(k)} & & & \end{bmatrix} \begin{bmatrix} v_{1,n+1} \\ v_{2,n+1} \\ v_{3,n+1} \\ v_{1,a} \\ v_{2,a} \\ v_{3,a} \end{bmatrix} = \begin{bmatrix} -I_{d2}^{(k)} \\ I_{d2}^{(k)} \\ \\ -I_{d1}^{(k)} \\ I_{d1}^{(k)} \end{bmatrix} \quad (8)$$

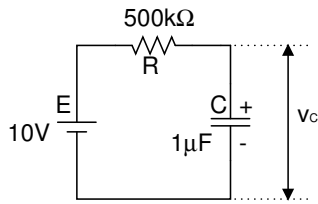


Figure 4: RC circuit

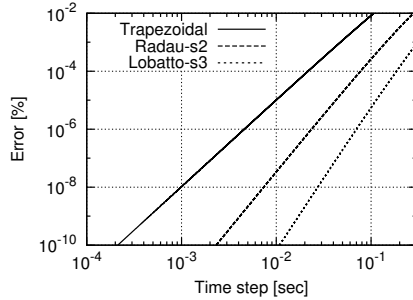


Figure 5: Error of each numerical integration

6. Comparison of computational costs

The waveform of simulation results using IRK methods for the diode circuit in Fig. 6 are shown in Fig. 7. We compare the computational costs between IRK methods and the trapezoidal method. Figure 8 shows the calculation time of two methods, where the horizontal axis is the time step of numerical integration and vertical axis is calculation time for three thousand periods. The straight line shows Lobatto-s3 and the dotted line shows the trapezoidal method. According to the graph, the computational costs of two methods become the same if Lobatto-s3 uses about 3.9 times the step size as the trapezoidal method. We can get much higher accuracy than the trapezoidal method, even if we use 3.9 times the step size as large as the trapezoidal, or more since Lobatto-s3 is the 4th order numerical integration.

7. Conclusion

This paper described the technique for implementing Radau-s2 and Lobatto-s3 of IRK formulas, which are A-stable and have high orders, as numerical integration into circuit simulators. Equivalent circuits at discrete time for capacitors and diodes were proposed. The equivalent circuit at one intermediate time was needed and the circuit size became twice as large as the trapezoidal method in Radau-s2 and Lobatto-s3. This problem was solved by using larger time step for the numerical integration, since the order of the numerical integration was high.

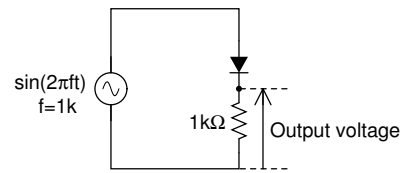


Figure 6: Diode circuit

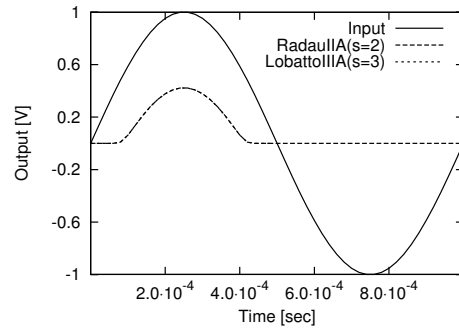


Figure 7: Waveform of the output

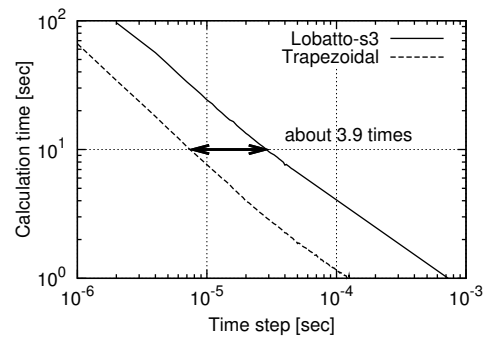


Figure 8: Comparison of computational costs

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