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A Fixed Point Theorem in Weak Topology for Successively Recurrent System of Fuzzy-Set-Valued Nonlinear Mapping Equations and Its Application

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Abstract—Let us introduce n (≥ 2) nonlinear mappings $f_i (i = 1, \cdots, n)$ defined on reflexive real Banach spaces X_{i-1} and let $f_i : X_{i-1} \to Y_i$ (a Banach space) be completely continuous on bounded convex closed subsets $X^{(0)}_{i-1} \subset X_{i-1}$. Moreover, let us introduce n fuzzy-set-valued nonlinear mappings $F_i : X_{i-1} \times Y_i \to$ {a family of all non-empty closed compact fuzzy subsets of X_i}.

Here, by introducing arbitrary constant $\beta_i \in (0, 1]$, for every integer i ($i = 1, \cdots, n \equiv 0$), separately, we have a fixed point theorem on the recurrent system of β_i-level fuzzy-set-valued mapping equations: $x_i \in F_{i\beta_i}(x_{i-1}, f_i(x_{i-1}))$, ($i = 1, \cdots, n \equiv 0$), where the fuzzy set F_i is characterized by a membership function $\mu_{F_i}(x_i) : X_i \to [0, 1]$, and the β_i-level set $F_{i\beta_i}$ of the fuzzy set F_i is defined as $F_{i\beta_i} \triangleq \{\xi_i \in X_i \mid \mu_{F_i}(\xi_i) \geq \beta_i\}$, for any constant $\beta_i \in (0, 1]$.

This theorem can be applied immediately to discussion for characteristics of ring nonlinear network systems disturbed by undesirable uncertain fluctuations and to extremely fine estimation of available behaviors of those disturbed systems. In this paper, the mathematical situation and proof are discussed, in weak topology.

1. Introduction

In general, by the set-valued mapping F defined on a Banach space X is meant a correspondence in which a set $F(x)$ is specified in correspondence to any point x in X. In particular, when $F(X) \subset X$, and if there exists a point x^* such that $x^* \in F(x^*)$, x^* is called a fixed point of F [1].

For an original completely continuous nonlinear mapping $f : X \to Y$ (a Banach space), the author gave some types of discussions of uncertain fluctuation problems, giving a composition type of fixed point theorem: $x \in F(x, f(x))$ concerning to a set-valued nonlinear mapping $F : X \times Y \to$ {the family of all non-empty closed compact subset of X} [1, 2, 3]. Thereafter, the author gave complex types of fixed point theorems for the system of set-valued nonlinear mapping equations, suitable to discussion for uncertain fluctuation problems of practical complex nonlinear systems.[4],[5]

In complex large-scale, so called as multi-media, network systems, the successively recurrent circular connectin of channels often plays an important role as a typical element network of local area networks(LAN). However, in large-scale circular networks, whenever some undesirable fluctuations are induced into their element channels, even if their influences are originally small to every element channel itself, total affections may be successively accumulated step by step, and as results, may grow so serious that the network itself becomes useless. Therefore, we must carefully evaluate and control those fluctuations such that overall behaviors of respective channel outputs can be led, in real time, into "available" or "tolerable" regions assigned in advance.

In such a reason, the author presented fixed point theorems for a successively recurrent system of set-valued mapping equations, and refered to available behaviors of signals to be appeared in every portion of succesively recurrent circular channels disturbed by undesirable fluctuations [6]. Moreover, by introducing the concept of fuzzy-set-valued mapping [7], as typical examples of much more refined theories, the author presented fixed point theorems for recurrent systems of fuzzy-set-valued nonlinear mapping equations,and referred to available behaviors of signals that appear in every portion of practical recurrent ring nonlinear network systems [8]. This paper is its more-refined theory of the similar problem under more wide conditions, through a precise deduction by introducing th fuzzy-set-valued mapping in weak topology.

2. Fuzzy Set and Fuzzy-Set-Valued Mapping

First of all, let us consider a family of all fuzzy sets originally introduced by Zadeh [7] in a Banach space X, and let any fuzzy set A be characterized by a membership function $\mu_A(x) : X \to [0, 1]$. Now, we can consider an α-level set A_α of the fuzzy set A as $A_\alpha \triangleq \{\xi \in X | \mu_A(\xi) \geq \alpha\}$, for any constant $\alpha \in (0, 1]$. The fuzzy set A is called compact, if all α-level sets are compact for arbitrary $\alpha \in (0, 1]$.

A fuzzy-set-valued mapping G from X into X is defined by $G : X \to \mathcal{F}(X)$, where $\mathcal{F}(X)$ is a family of all non-empty , bounded and closed fuzzy sets in X. If a point $x \in X$ is mapped to a fuzzy set $G(x)$, the membership function of $G(x)$ at the point $\xi \in X$ is represented by $\mu_{G(x)}(\xi)$.

For convenience, let us introduce a useful notation: for an arbitrarily specified constant $\beta \in (0, 1]$, a point x belongs to the β-level set A_β of the fuzzy set A: $x \in A_\beta \triangleq \{\xi \in X \mid$

$\mu_A(\xi) \geq \beta\}$ is denoted by $x \in_\beta A$ [9].

Here, let us introduce a new concept of β-level fixed point: for the fuzzy set $G(x)$, if there exists a point x^* such that $x^* \in_\beta G(x^*)$, then x^* is called β-level fixed point of the fuzzy-set-valued mapping G [9].

Now, let us remember that we have introduced a new metric into the space of fuzzy sets [9].

Definition 1 *Let us consider a Banach space X. For any fixed constant $\beta \in (0, 1]$, the β-level metric ρ_β between a point $x \in X$ and a fuzzy set A is defined as follows:*

$$\rho_\beta(x, A) \triangleq \inf_{\beta \leq \alpha \leq 1} d_\alpha(x, A), \tag{1}$$

where

$$d_\alpha(x, A) \triangleq \begin{cases} \inf\limits_{y \in A_\alpha} \|x - y\| & \text{if } \alpha \leq \alpha_A, \\ \inf\limits_{y \in A_{\alpha_A}} \|x - y\| & \text{if } \alpha > \alpha_A. \end{cases} \tag{2}$$

Here, $\alpha_A \triangleq \sup_{x \in X} \mu_A(x)$. And also, for any fixed constant $\beta \in (0, 1]$, by means of the Hausdorff metric d_H, the β-level metric $\mathcal{H}_\beta$ between two fuzzy sets A and B is introduced as follows:

$$\mathcal{H}_\beta(A, B) \triangleq \sup_{\beta \leq \alpha \leq 1} D_\alpha(A, B), \tag{3}$$

where D_α is defined as

$$D_\alpha(A, B) \triangleq \begin{cases} d_H(A_\alpha, B_\alpha) \\ \qquad \text{if } \alpha \leq \min\{\alpha_A, \alpha_B\}, \\ d_H(A_{\alpha_A}, B_\alpha) \\ \qquad \text{if } \alpha_A < \alpha \leq \alpha_B, \\ d_H(A_\alpha, B_{\alpha_B}) \\ \qquad \text{if } \alpha_A \geq \alpha > \alpha_B, \\ d_H(A_{\alpha_A}, B_{\alpha_B}) \\ \qquad \text{if } \alpha > \max\{\alpha_A, \alpha_B\}. \end{cases} \tag{4}$$

Here, $\alpha_B \triangleq \sup_{x \in X} \mu_B(x)$ and the Hausdorff metric d_H between two sets S_1 and S_2 is defined by

$$d_H(S_1, S_2) \triangleq \max\{\sup\{d(x_1, S_2) | x_1 \in S_1\}, \sup\{d(x_2, S_1) | x_2 \in S_2\}\}, \tag{5}$$

where $d(x, S) \triangleq \inf\{\|x - y\| \mid y \in S\}$ is the distance between a point x and a set S.

In order to give a new methodology for the discussion more sophisticated than the one by usual crisp set-valued mappings, the author presented mathematical theories based on the concept of β-level fixed point, by establishing fixed point theorems for β-level fuzzy-set-valued nonlinear mappings which describe detailed characteristics of such fuzzy-set-valued nonlinear mapping equations, for every level $\beta \in (0, 1]$ [8, 9]. Particularly, by introducing arbitrary constant $\beta_i \in (0, 1]$, for every integer i ($i = 1, \cdots, n \equiv 0$), separately, the author gave a refined theory suitable to precise discussion for characteristics of disturbed ring nonlinear network systems [8].

3. Fixed Point Theorem in Weak Topology for Successively Recurrent System of Fuzzy-Set-Valued Mapping Equations

Here, we will present the fixed point theorem in weak topology for successively recurrent system of fuzzy-set-valued mapping equations.

For the first step, let us introduce reflexive real Banach spaces X_i ($i = 1, \ldots, n \equiv 0$), in which the norm is represented by $\|\cdot\|$, and also, let us define there non-empty bounded closed convex subsets $X_i^{(0)}$ ($i = 1, \ldots, n \equiv 0$). Let X_i' be the dual space of X_i and let us introduce a weak topology $\sigma\left(X_i, X_i'\right)$ into X_i. Then, X_i is locally convex topological linear space,and therefore, $X_i^{(0)}$ is weakly closed and weakly compact. Further, let us consider another real Banach space Y_i ($i = 1, \ldots, n \equiv 0$) in which the norm is represented by $\|\cdot\|$.

Now, let us introduce n (≥ 2) mappings f_i ($i = 1, \ldots, n \equiv 0$) defined on X_{i-1}, respectively, and let $f_i : X_{i-1} \to Y_i$ be completely continuous on bounded convex closed subset $X_{i-1}^{(0)} \subset X_{i-1}$ ($i = 1, \ldots, n \equiv 0$).

Moreover, let us introduce n fuzzy-set-valued mappings $F_i : X_{i-1} \times Y_i \to \mathcal{F}_c(X_i)$ (the family of all non-empty compact fuzzy subsets of X_i) ($i = 1, \ldots, n \equiv 0$).

Here, we can recognize that for any $x_{i-1} \in X_{i-1}^{(0)}$ and $f_i(x_{i-1}) \in Y_i$, we have $F_i^{(0)}(x_{i-1}; f_i(x_{i-1})) \triangleq F_i^{(0)}(x_{i-1}; f_i(x_{i-1})) \cap X_i^{(0)} \neq \phi$, and moreover, there exist β_i-level projection points $\tilde{x}_i'$ of arbitry point $x_i' \in X_i^{(0)}$ upon the fuzzy set $F_i^{(0)}(x_{i-1}; f_i(x_{i-1}))$ such that

$$\left\|\tilde{x}_i{}' - x_i'\right\| = \min\{\left\|x_i' - z_i\right\| \mid z_i \in_{\beta_i} F_i^{(0)}(x_{i-1}; f_i(x_{i-1}))\}. \tag{6}$$

Now, let us introduce a series of assumptions:

Assumption 1 *Let the mapping $f_i : X_{i-1}^{(0)} \to f_i(X_{i-1}^{(0)}) \subset Y_i$ be completely continuous in the sense that when a weakly convergent net $\{x_i^\nu\}$ ($\nu \in J$: a directive set) weakly converges to $\overline{x}_i$, then the sequence $\{f_i(x_{i-1}^\nu)\}$ has a subsequence which strongly converges to $f_i(\overline{x}_{i-1})$ in Y_i.*

Assumption 2 *Let the fuzzy-set-valued mapping $F_i : X_{i-1}^{(0)} \times Y_i \to \mathcal{F}_c(X_i)$ (the family of all non-empty compact fuzzy subsets of X_i) satisfies the following Lipschits condition with respect to the β_i-level metric $\mathcal{H}_{\beta_i}$, that is, there are two kinds of constants $a_i \equiv a_i(\beta_i) > 0$ and $b_i \equiv b_i(\beta_i) > 0$ such that for any $x_{i-1}^{(1)}, x_{i-1}^{(2)} \in X_{i-1}$ and for any $y_i^{(1)}, y_i^{(2)} \in Y_i$, F_i satisfies the inequality*

$$\begin{aligned} &\mathcal{H}_{\beta_i}\left(F_i(x_{i-1}^{(1)}; y_i^{(1)}), F_i(x_{i-1}^{(2)}; y_i^{(2)})\right) \\ &\leq a_i \cdot \left\|x_{i-1}^{(1)} - x_{i-1}^{(2)}\right\| + b_i \cdot \left\|y_i^{(1)} - y_i^{(2)}\right\|, \end{aligned} \tag{7}$$

where Lipschitz constants a_i ($i = 1, \ldots, n \equiv 0$) are confined by

$$0 < a_1 \cdot a_2 \cdot \ldots \cdot a_n < 1. \tag{8}$$

Then, under these preparations, we have an important lemma on the successively recurrent system of fuzzy-set-valued mapping equations:

$$x_i \in_{\beta_i} F_i(x_{i-1};\, f_i(x_{i-1})),\ (i = 1, \ldots, n \equiv 0). \tag{9}$$

Lemma 1 *For all i ($i = 1, \ldots, n \equiv 0$), let us adopt arbitrary points $x_i^{(0)} \in X_i^{(0)}$ and also fix all values of $f_i(x_{i-1}^{(0)})$ ($i = 1, \ldots, n \equiv 0$). Now, for every i, let us consider a sequence $\{x_i^{(\nu)}\}(\nu = 0, 1, 2, \ldots)$ starting from the above adopted point $x_i^{(0)}$, and with each $x_i^{(\nu)} \in X_i^{(0)}$ as the β_i-level projection point of $x_i^{(\nu-1)} \in X_i^{(0)}$ upon the fuzzy set $F_i(x_{i-1}^{(\nu-1)};\, f_i(x_{i-1}^{(0)}))$, but for any number $m \geq 1$ ($m \leq n - i$), $x_{i+m}^{(\nu)} \in X_{i+m}^{(0)}$ is specified as the β_{i+m}-level projection point from $x_{i+m}^{(\nu-1)} \in X_{i+m}^{(0)}$ upon the fuzzy set $F_i(x_{i+m-1}^{(\nu)};\, f_{i+m}(x_{i+m-1}^{(0)}))$. Then, these sequences $\{x_i^{(\nu)}\}$ ($i = 1, \ldots n \equiv 0; \nu = 0, 1, 2, \ldots$) are Cauchy sequences, having their own limit points $\bar{x}_i \in X_i^{(0)}$, respectively, such that*

$$\bar{x}_i \in_{\beta_i} F_i^{(0)}(\bar{x}_{i-1};\, f_i(x_{i-1}^{(0)})),\ (i = 1, \ldots, n \equiv 0). \tag{10}$$

All limit points $\bar{x}_i$ ($i = 1, \ldots, n \equiv 0$) depend on their starting points $x_i^{(0)}$ and parameters $y_i^{(0)} \triangleq f_i(x_{i-1}^{(0)})$, respectively. These correspondences may be multi-valued, in general, and hence, can be represented by set-valued continuous mappings defined on each domain:

$$\bar{x}_i \in H_i(x_i^{(0)}, y_i^{(0)}),\ (i = 1, \ldots, n \equiv 0). \tag{11}$$

If these mappings have fixed points x_i^* in respective domain, or in respective bounded convex closed subsets $X_i^{(0)}$: i.e.,

$$x_i^* \in H_i(x_i^*; y_i^*) \in X_i^{(0)},\ (i = 1, \ldots, n \equiv 0), \tag{12}$$

where $y_i^* \triangleq f_i(x_{i-1}^*)$, these relations imply that

$$x_i^* \in_{\beta_i} F_i^{(0)}(x_{i-1}^*;\, f_i(x_{i-1}^*)),\ (i = 1, \ldots, n \equiv 0). \tag{13}$$

This result means that the solution set $\{x_i^*\}$ ($i = 1, \ldots, n \equiv 0$) of the system of set-valued nonlinear mapping equations (9) can be obtained in connection with the set of limit points $\{\bar{x}_i\}$ ($i = 1, \ldots, n \equiv 0$) of Cauchy sequences $\{x_i^{(\nu)}\}$ ($i = 1, \ldots, n \equiv 0; \nu = 0, 1, 2, \ldots$).

Here, in order to verify the existance of the fixed point x_i^* of H_i, from the standpoint refined in the weak topology, now, it is very convenient to apply the well-known fixed point theorem for set-valued mapping:

Lemma 2 (Ky Fan [10]) *Let X_i be a locally convex topological linear space, and $X_i^{(0)}$ be a non-empty convex compact subset of X_i. Let $\mathcal{H}_c(X_i^{(0)})$ be the family of all non-empty closed convex subsets of $X_i^{(0)}$. Then, for upper semi-continuous set-valued mapping $H_i : X_i^{(0)} \to \mathcal{H}_c(X_i^{(0)})$, there exists a fixed point $x_i^* \in X_i^{(0)}$ such that $x_i^* \in H_i(x_i^*)$.*

In order to apply this lemma to our problem, we can verify that the above-defined set-valued mapping $H_i(x_i) \equiv H_i(x_i; y_i^{(0)})$ is upper semicontinuous, and its range is closed and convex.

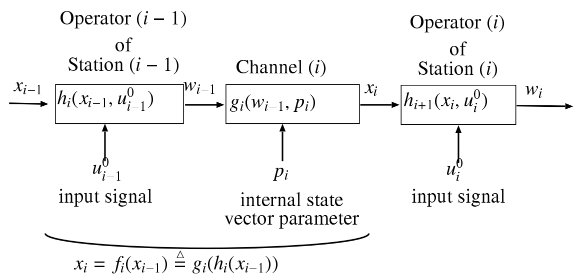
As a result, we have the theorem:

Theorem 1 *Let X_i be a reflexive real Banach space, and $X_i^{(0)}$ be a non-empty bounded closed convex subset of X_i. By the dual space X_i', let us introduce a weak topology $\sigma(X_i, X_i')$ into X_i. Let f_i and F_i be deterministic and fuzzy-set-valued mappings, respectively, which satisfy Assumptions 1 and 2. Then, we have a Cauchy sequence $\{x_i^\nu\} \subset X_i^{(0)}$ ($\nu = 0, 1, \ldots$), introduced by the successive procedure in Lemma 1. This sequence has a set of limit points $\{\bar{x}_i\}$, and we can define a set-valued mapping H_i by the correspondence from the arbitrary starting point $z_i^{(0)} \equiv x_i \in X_i^{(0)}$ to the set of limit points $\{\bar{x}_i\}$ in $X_i^{(0)}$: $\bar{x}_i \in H_i(x_i)$. This set-valued mapping H_i has a fixed point x_i^* in $X_i^{(0)}$, which is, in turn, the solution of the system of set-valued mapping equations (9).*

4. An Application to Recurrent Ring Nonlinear Network Systems with Undesirable Fluctuations

The fixed point theorem here-derived can be applied immediately to analysis for characteristics of recurrent ring nonlinear network systems with undesirable fluctuations and to fine evaluation of available behaviors of those disturbed systems.

Let us consider a recurrent ring nonlinear network system which consists of n stations and n unilateral nonlinear channels. The station (i) is operated by the operator (i), ($i = 1, \cdots, n \equiv 0$). For instance, in consideration of the output signal x_{i-1} of the channel ($i-1$), the operator ($i-1$), at the station ($i-1$), operates his own input signal u_{i-1}^0 and gives the input signal w_{i-1} into the channel (i). The channel (i) transfers this input signal w_{i-1} to the output side as an input signal x_i of the station (i). The function of the operator ($i-1$) is described as



$$w_{i-1} = h_i(x_{i-1}, u_{i-1}^0), \tag{14}$$

and the nonlinear function of the channel (i) is described as

$$x_i = g_i(w_{i-1}, p_i), \tag{15}$$

where p_i denotes the internal state vector parameter of channel (i), representative of whole internal structures and parameters. For fixed u_{i-1}^0 and p_i, functions h_i and g_i can be abbreviated as $h_i(x_{i-1})$ and $g_i(w_{i-1})$, respectively. Incidentally, we denote

$$f_i(x_{i-1}) \triangleq g_i(h_i(x_{i-1})). \tag{16}$$

As we previously pointed out, undesirable fluctuations are often induced into ill-conditioned communication channels. In fact, in our ring network system, output signals x_i sometimes undergo undesirable uncertain fluctuations forming closed subsets $G_i(h_i(x_{i-1}), f_i(x_{i-1}))$ in each space X_i, respectively. So, for such a disturbed output signal x_i, we can define a new set-valued nonlinear mapping:

$$F_i(x_{i-1}, f_i(x_{i-1})) \triangleq G_i(h_i(x_{i-1}), f_i(x_{i-1})). \tag{17}$$

Thus, the behavior of the channel (i) can be described in the form:

$$x_i \in F_i(x_{i-1}, f_i(x_{i-1})). \tag{18}$$

Originally, these sets are crisp. However, in order to introduce more fine estimation into these resultant fluctuation sets, here we can reconsider anew these sets F_i as fuzzy sets. Then, let us replace the above described crisp sets $F_i(x_{i-1}, f_i(x_{i-1}))$ by fuzzy sets with same notations, accompanied with suitable membership functions $\mu_{F_i}(\xi_i), \xi_i \in X_i$, which should be properly introduced corresponding to conscious planning for the fine evaluation of resultant fluctuations themselves.

Now, for any fixed constant $\beta_i \in (0, 1]$ different for every channel (i), separately, then, we can accomplish fine estimation of fuzzy set $F_i(x_i, f_i(x_{i-1}))$, by using individual β_i-level ($i = 1, \cdots, n \equiv 0$), respectively. So, for every fixed constant $\beta_i \in (0, 1]$, we can introduce a new system of β_i-level fuzzy-set-valued nonlinear mapping equations:

$$x_i \in_{\beta_i} F_i(x_{i-1}, f_i(x_{i-1})),\ (i = 1, 2, \cdots, n \equiv 0). \tag{19}$$

If there exists a set of β_i-level fixed points $\{x_i^*\}$ in $X_i^{(0)}$ ($i = 1, \cdots, n \equiv 0$), which satisfy the system of β_i-level fuzzy-set-valued mapping equations (19), each x_i^* can be considered as a β_i-level likelihood output signal of channel (i), with internal state p_i being affected by uncertain fuzzy fluctuation, for fixed input signal u_{i-1}^0 of station $(i-1)$ ($i = 1, \cdots, n \equiv 0$). Here, this β_i-level likelihood signal x_i^* can be found in a closed domain in which membership function $\mu_{F_i(x_{i-1}^*, y_i^*)}(\xi_i)$ has value larger than or equal to β_i, individually.

Thus, the fluctuation analysis of this type of recurrent ring networks of unilateral nonlinear channels, undergone by undesirable uncertain fluctuations, can be successfully accomplished at extremely fine-level of estimation, by immediate application of the here-presented fixed point theorem in weak topology for system of β_i-level fuzzy-set-valued nonlinear mappings, with consciously selected values of parameters β_i ($i = 1, \cdots, n \equiv 0$).

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