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Perturbation theory for unstable periodic orbits in chaotic dynamical systems

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Abstract—Unstable periodic orbits (UPOs) in chaotic attractors dominate statistical properties of chaos, and understanding behavior of the UPOs is quite important for understanding chaos. In this research, we study the response of the UPOs to external forces or small parameter changes by applying perturbation theory. We show that the shift of the trajectories of the UPOs can be approximated by the perturbation expansion despite the difficulty to track the small deviation from the periodic orbit due to positive Floquet exponents. We applied this method to some UPOs of the logistic map. We found that the lowest order perturbation theory predicts the shift of the invariant measure under parameter change. This result can be a basis of the future application of the perturbation theory of chaos, which enables us to predict its response.

1. Introduction

Perturbation theory plays an important role in nonlinear dynamics. It has been applied in various fields such as synchronization [1], pattern formation [2], etc. Particularly, phase reduction [3] is a perturbation theory of stable periodic orbits focusing on the zero Floquet eigenvalue.

It is useful to understand responses of chaotic systems to perturbation for controlling or forecasting them. Here perturbation includes external forcing, coupling to another chaotic oscillator, change of parameter of the system, etc. However, it is difficult to perturbation theory for chaotic systems because of the orbital instability.

Unstable periodic orbits (UPOs) play important roles to characterize chaos [4]. Cycle expansion formalisms was proposed to express statistical quantities of chaos in terms of sum over periodic orbits in a chaotic attractor [5, 6, 7, 8]. Recently, UPOs which have similar statistical property to that of plane Couette flow [9] and the shell model [10] were numerically obtained. Emergence of periodic windows of chaotic systems can be predicted from local manifold structures of UPOs [11]. UPOs also attract much interest in terms of chaos control [12, 13]. Methods of finding out UPOs from time series [14, 15, 16] were proposed.

If the system is structurally stable, we expect that the UPO exists under the perturbation close to the orbit of the unperturbed UPO. Therefore, it would be possible to predict change of the orbit and other quantities for the perturbed UPO using the perturbation theory. In other words, it is important to uncover the response to perturbation of an UPO. In this paper, we show that it is possible to apply the perturbation theory to UPOs in chaotic attractors.

This paper is organized as follows. In Sec. 2, the formalism of the perturbation theory is presented. Response to perturbations is decomposed into modes of the Floquet matrix of the UPO and expanded in terms of the perturbation parameter. This formalism is applied in the logistic map and numerically validated in Sec. 3. Summary and discussions are given in the last section.

2. Formalism

In this section, we introduce our framework to apply the perturbation theory to UPOs of chaotic maps. We consider a discrete time evolution equation

$$\mathbf{X}(\varepsilon; n+1) \quad = \mathbf{F}[\mathbf{X}(\varepsilon; n)] + \varepsilon \mathbf{p}[n, \mathbf{X}(\varepsilon; n)]. \tag{1}$$

Here, the second term represents the perturbation, and ε denotes the perturbation parameter. The perturbation vanishes for $\varepsilon = 0$. $\mathbf{X}(\varepsilon; n)$ denotes the dynamical variable $\mathbf{X} \in \mathbf{R}^d$ at time step n with the perturbation parameter ε. We assume that this system has a chaotic attractor in the absence of the perturbation $\varepsilon = 0$.

We consider an UPO of the unperturbed state with period N, $\mathbf{X}_0(0; n)$, which satisfies

$$\mathbf{X}_0(0; N+1) \quad = \quad \mathbf{X}_0(0; 1). \tag{2}$$

Our goal is to estimate the trajectory of this UPO in the presence of the perturbation $\varepsilon\mathbf{p}$ using the information of the unperturbed state $\mathbf{X}_0(0; n)$. We further assume that this UPO exists and deforms continuously in ε. Then one can expand with the power series in ε as

$$\mathbf{X}_0(\varepsilon; n) = \mathbf{X}_0(0; n) + \sum_{\nu=1}^{\infty} \varepsilon^{\nu} \mathbf{x}_{\nu}(n). \tag{3}$$

Substituting this expansion to Eq. (1), and collecting terms of ε^{ν}, one can solve these equations sequentially from the lowest order equation. One can easily verify that the zeroth order equation in $O(\varepsilon^0)$ is

$$\mathbf{X}(0; n+1) = \mathbf{F}[\mathbf{X}(0; n)], \tag{4}$$

which is identical to the unperturbed equation. The non-trivial relationship appears in the first order $O(\varepsilon^1)$, which can be written as

$$\mathbf{x}_1(n+1) = \hat{L}_0[\mathbf{X}_0(n)]\mathbf{x}_1(n) + \mathbf{p}[n, \mathbf{X}_0(0; n)], \tag{5}$$

where $\hat{L}_0[\mathbf{X}_0(n)]$ denotes the Jacobian of the time evolution function $\mathbf{F}[\mathbf{X}_0(n)]$. Note that the Jacobian $\hat{L}_0$ is a function of n through $\mathbf{X}_0(n)$. Hence, $\hat{L}_0[\mathbf{X}_0(n)]$ and $\mathbf{p}[n, \mathbf{X}_0(0; n)]$ are respectively written as $\hat{L}_0(n)$ and $\mathbf{p}(n)$ in the following. Note also that the Jacobian is periodic, i.e., $\hat{L}_0(N) = \hat{L}_0(0)$ holds.

Solution of Eq. (5) can be written in terms of the Floquet matrix. Let us consider the linearized equation of Eq. (1):

$$\mathbf{u}(n+1) = \hat{L}_0(n)\mathbf{u}(n). \tag{6}$$

Since $\hat{L}_0(n)$ is periodic, the solution of this equation is given as

$$\mathbf{u}(n) = \hat{S}_0(n)\hat{\Lambda}_0^n \mathbf{u}(0), \tag{7}$$

according to the Floquet theorem. Here, $\hat{S}_0(n)$ is assumed to be periodic, in particular that $\hat{S}_0(N) = \hat{S}_0(0) = \hat{I}_d$ holds, where $\hat{I}_d$ denotes the d dimensional identity matrix. According to this periodicity of S_0, $\hat{\Lambda}_0^N$ determines the exponential growth or decay of the initial condition $\mathbf{u}(0)$ for one period:

$$\mathbf{u}(N) = \Lambda_0^N \mathbf{u}(0). \tag{8}$$

The largest eigenvalue of $\hat{\Lambda}_0^N$, λ_1^N, gives the "average" growth of one time step. Substituting Eq. (7) into Eq. (6), one obtains the equality between $\hat{S}_0(n)$, $\hat{\Lambda}_0$, and $\hat{L}_0(n)$ as

$$\hat{L}_0(n) = \hat{S}_0(n+1)\hat{\Lambda}_0\hat{S}_0^{-1}(n). \tag{9}$$

This equality holds for arbitrary initial condition $\mathbf{u}(0)$ for $\varepsilon = 0$, as long as the linearization approximation is valid. The Jacobian $\hat{L}_0(n)$ in Eq. (5) is replaced and we obtain

$$\mathbf{x}_1(n+1) = \hat{S}_0(n+1)\hat{\Lambda}_0\hat{S}_0^{-1}(n)\mathbf{x}_1(n) + \mathbf{p}(n). \tag{10}$$

In the following we obtain the first order perturbation using Eq. (10). We introduce new variables

$$\overline{\mathbf{x}}_1(n) = \hat{S}_0^{-1}(n)\mathbf{x}_1(n), \tag{11}$$

$$\overline{\mathbf{p}}(n) = \hat{S}_0^{-1}(n+1)\mathbf{p}(n). \tag{12}$$

Substituting them into Eq. (10), one can easily verify that the evolution equation for $\overline{\mathbf{x}}_1$ is given as

$$\overline{\mathbf{x}}_1(n+1) = \hat{\Lambda}_0\overline{\mathbf{x}}_1(n) + \overline{\mathbf{p}}(n). \tag{13}$$

Since $\hat{\Lambda}_0$ does not depend on time, the above equation is solved as

$$\overline{\mathbf{x}}_1(n) = \hat{\Lambda}_0^n\overline{\mathbf{x}}_1(0) + \sum_{k=0}^{n-1} \hat{\Lambda}_0^k \overline{\mathbf{p}}(n-k-1). \tag{14}$$

We then expand this equation in terms of $\mathbf{v}_j$, the right eigenvector of $\hat{\Lambda}_0$ for the eigenvalue λ_j, as

$$\overline{\mathbf{x}}_1(n) = \sum_{j=1}^{d} c_{j1}(n)\mathbf{v}_j, \tag{15}$$

$$\overline{\mathbf{p}}(n) = \sum_{j=1}^{d} p_j(n)\mathbf{v}_j. \tag{16}$$

Once the coefficients c_{j1}'s are obtained, the first order perturbation is achieved as

$$\mathbf{x}_1(n) = \sum_{j=1}^{d} c_{j1}(n)\hat{S}_0(n)\mathbf{v}_j, \tag{17}$$

from Eq. (11). Multiplying the left eigenvector $\mathbf{v}_j^*$, which satisfies the normalization condition $(\mathbf{v}_j^*)^T\mathbf{v}_l = \delta_{jl}$, to Eq. (14) from the left, we have

$$c_{j1}(n) = \lambda_j^n c_{j1}(0) + \sum_{k=0}^{n-1} \lambda_j^k p_j(n-k-1). \tag{18}$$

This is the coefficient for jth mode of the first order perturbation.

The final step is to apply Eq. (18) to the UPOs and obtain the perturbed periodic orbit. If the UPO under consideration remains in the presence of a small perturbation, all perturbation coefficients must satisfy the periodicity. Therefore, the condition

$$c_{j1}(N) = c_{j1}(0) \tag{19}$$

is required. Substituting this condition to Eq. (18) and taking $n = N$, one obtains

$$c_{j1}(0) = \frac{1}{1-\lambda_j^N}\left[\sum_{k=0}^{N-1} \lambda_j^k p_j(N-k-1)\right]. \tag{20}$$

To the first order, Eq. (9) is rewritten as $[\hat{S}_0(n+1) + \varepsilon\hat{S}_1(n+1)](\hat{\Lambda}_0 + \varepsilon\hat{\Lambda}_1) = [\hat{L}_0(n) + \varepsilon\hat{L}_1(n)][\hat{S}_0(n) + \varepsilon\hat{S}_1(n)]$. Note the periodicity condition $\hat{S}_1(0) = \hat{S}_1(N) = \hat{0}$. We can derive the first order perturbation for the Floquet exponent from this equation. The equation for the $O(\varepsilon^1)$ terms is given as

$$\hat{S}_1(n+1) = \hat{L}_0(n)\hat{S}_1(n)\hat{\Lambda}_0^{-1} + \hat{P}(n)\hat{\Lambda}_0^{-1}. \tag{21}$$

where $\hat{P}(n) = \hat{L}_1(n)\hat{S}_0(n) - \hat{S}_0(n+1)\hat{\Lambda}_1$. This equation can be solved in terms of $\hat{S}_1(n)$ as

$$\hat{S}_1(n) = \hat{S}_0(n)\left(\sum_{k=0}^{n-1} \hat{\Lambda}_0^k \hat{S}_0^{-1}(n-k)\hat{P}(n-k-1)\hat{\Lambda}_0^{-k-1}\right). \quad (22)$$

Taking $n = N$ and using the periodicity condition $\hat{S}_0(0) = \hat{S}_0(N) = \hat{I}_d$ and $\hat{S}_1(0) = \hat{S}_1(N) = \hat{0}$, the identity $\hat{P}(N) = \hat{0}$ must be satisfied. Thus, we obtain

$$\Lambda_1 = \sum_{n=0}^{N-1} \hat{S}_0^{-1}(n+1)\hat{L}_1(n)\hat{S}_0(n). \quad (23)$$

Eigenvalues of Λ_1 is given as

$$\lambda_l^{(1)} = \mathbf{v}_l^* \hat{\Lambda}_1 \mathbf{v}_l \quad (24)$$
$$= \sum_{n=1}^{N} \mathbf{v}_l^* \hat{S}_0^{-1}(n+1)\hat{L}_1(n)\hat{S}_0(n)\mathbf{v}_l. \quad (25)$$

3. Numerical example: Logistic map

We apply this formalism to UPOs of the logistic map

$$X(\varepsilon; n+1) = f_a(\varepsilon; X(\varepsilon; n)) \quad (26)$$
$$= (a+\varepsilon)X(\varepsilon; n)[1 - X(\varepsilon; n)], \quad (27)$$

and see how the perturbation theory can predict the behavior of UPOs. The dynamical variable X is a scalar for the case of one dimensional map. Here, the unperturbed state is the logistic map with $a = 3.9$, and we regard the small change of the parameter a as the perturbation. We want to predict an UPO for $a + \varepsilon$ using the UPO for a and the perturbation theory. The trajectories of the chaos and some UPOs are depicted in Fig. 1. In this case, the perturbation is $p(n) = X(0; n)[1 - X(0; n)]$.

The lowest order perturbation is achieved from Eqs. (3) and (17) as

$$X_0^{\text{th}}(\varepsilon; n) = X_0(0; n) + \varepsilon c_1(n) S_0(n) + O(\varepsilon^2). \quad (28)$$

Note that subscript j specifying the eigenvalue in Eq. (17) does not appear because we study a one dimensional system. The eigenvalue λ coincides with

$$\lambda^N = \prod_{n=1}^{N} |f_a'(0; X_0(0; n))|, \quad (29)$$

in this case. The eigenvector $\mathbf{v}$ is scholar, and we take $v = 1$ taking into account the normalization condition. For $n = 0$, it is easy to obtain

$$X_0^{\text{th}}(\varepsilon; 0) = X_0(0; 0) + \varepsilon \frac{1}{1-\lambda^N}\left[\sum_{k=0}^{N-1} \lambda^k p(N-k-1)\right] + O(\varepsilon^2), \quad (30)$$

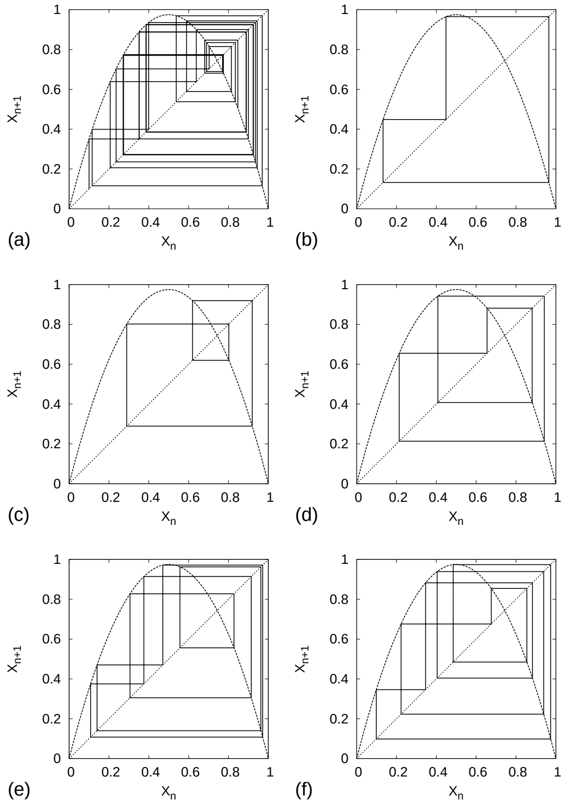

Figure 1: Trajectories of chaos of the logistic map for $a = 3.9$ (a) and UPOs of period 3 (b), period 4 (c), period 5 (d). Two different period 10 UPOs are depicted in panels (e) and (f). In panel (a), aperiodic chaotic orbit for 30 steps is presented.

from Eq. (20).

We numerically obtained some UPOs for $a = 3.9$, and $\varepsilon = 0$ and $\varepsilon = 0.01$. The perturbation is calculated using the UPOs for $\varepsilon = 0$, and the orbits for $\varepsilon = 0.01$ are predicted. Comparison is presented in Tab. 1. The numerical results and the predictions are in good agreement especially for UPOs with small period. For UPOs of longer period, the agreement tends to worsen. This result suggests that the higher order perturbations have to be considered for longer periodic orbits.

4. Summary

In this paper, we studied the perturbation expansion of the UPOs of chaotic maps. We derived the lowest order equation and applied this formalism to the UPOs of the logistic map, and found that the theory can predict the shift of the trajectories of such UPOs, especially if the period of the orbit is short. Although the precision of the approximation tends to be worse for longer period UPOs, shorter period UPOs contribute to the chaotic dynamics more, and our method would be applicable for predicting the response

Table 1: Comparison of the unstable periodic orbits for $a = 3.9$, $\varepsilon = 0$ and for $a = 3.9$, $\varepsilon = 0.01$, and ones predicted by the perturbation theory, $X_0^{\rm th}(0.01; 0)$. In the final column, the ratio $X_0^{\rm th}(0.01; 0)/X_0(0.01; 0)$ is shown. Perfect agreement between theory and numerical solution gives ratio=1.

period	$X_0(0; 0)$	$X_0(0.01; 0)$	$X_0^{\rm th}(0.01; 0)$	ratio
3	0.130718	0.132652	0.132593	0.97
4	0.621347	0.619507	0.619521	0.999
5	0.655727	0.654960	0.654946	1.02
6	0.550977	0.546091	0.546282	0.96
10	0.461939	0.470205	0.469300	0.89
10	0.974317	0.974090	0.974095	0.98
10	0.963672	0.962648	0.962747	0.90

of the chaos.

There will be many possible application of this method, e.g., predicting the response of chaos incorporating with the cycle expansion [5, 6] or applying this method to single dominant UPO [9, 10].

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