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Rigorous Computation of the Monodromy and the Pruning Front of Dynamical Systems

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Abstract—The focus of this work is the interplay between two distinct areas of dynamical systems: one is the monodromy theory of polynomial maps with complex variables; the other is the pruning front theory, a generalization of the kneading invariant for unimodal maps.We prove that the dynamics of a real polynomial map is governed by the monodromy of the same map extended to complex parameter and phase spaces, provided some hyperbolicity conditions. As an application, we identify the pruning front of the Hénon map for some parameter values.

1. Hubbard's Conjectures

One of the motivations of this work is to give an answer to the conjecture of John Hubbard on the topology of hyperbolic horseshoe locus of the complex Hénon map

$$H_{a,c} : \mathbb{C}^2 \to \mathbb{C}^2 : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x^2 + c - ay \\ x \end{pmatrix}.$$

Here a and c are complex parameters.

Below we describe the conjecture following a formulation given by Bedford and Smillie [3].

Let us define the filled Julia set

$$K^{\mathbb{C}}_{a,c} := \{p \in \mathbb{C}^2 : \{H^n_{a,c}(p)\}_{n\in\mathbb{Z}} \text{ is bounded}\}$$

and its real slice $K^{\mathbb{R}}_{a,c} := K^{\mathbb{C}}_{a,c} \cap \mathbb{R}^2$. When the parameters a and c are both real, the real plane $\mathbb{R}^2 \subset \mathbb{C}^2$ is invariant.In this case, we call $H_{a,c}|_{\mathbb{R}^2} : \mathbb{R}^2 \to \mathbb{R}^2$ the *real* Hénon map.

Our primary interest is on the structure of the parameter space, especially on the topology of the set of parameters on which the map become a uniformly hyperbolic horseshoe. More precisely, we study the following sets:

$$\mathcal{H}^{\mathbb{C}} := \{(a, c) \in \mathbb{C}^2 : H_{a,c}|K^{\mathbb{C}}_{a,c} \text{ is a hyp. full horseshoe}\},$$
$$\mathcal{H}^{\mathbb{R}} := \{(a, c) \in \mathbb{R}^2 : H_{a,c}|K^{\mathbb{R}}_{a,c} \text{ is a hyp. full horseshoe}\}.$$

Here we mean by a hyperbolic full horseshoe an uniformly hyperbolic invariant set which is topologically conjugate to the full shift map σ defined on $\Sigma_2 = \{0, 1\}^{\mathbb{Z}}$.

A classical result of Devaney and Nitecki claims that

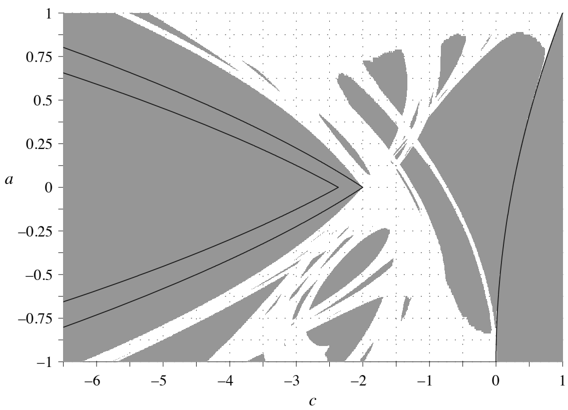
$$\mathsf{DN} := \{(a, c) \in \mathbb{R}^2 : c < -(5 + 2\sqrt{5})\,(|a| + 1)^2/4,\ a \neq 0\}$$

is contained in $\mathcal{H}^{\mathbb{R}}$. They also showed that the set

$$\mathsf{EMP} := \{(a, c) \in \mathbb{R}^2 : c > (a + 1)^2/4\}$$

consists of parameter values such that $K^{\mathbb{R}}_{a,c} = \emptyset$. Later, Hubbard and Oberste-Vorth investigated the Hénon map form the complex dynamical point of view, and improved the hyperbolicity criterion by showing that

$$\mathsf{HOV} := \{(a, c) \in \mathbb{C}^2 : |c| > 2(|a| + 1)^2,\ a \neq 0\} \subset \mathcal{H}^{\mathbb{C}}.$$



The figure above illustrates a subset of parameter values on which the chain recurrent set of the *real* Hénon map is uniformly hyperbolic (not necessarily a full horseshoe) [1]. Solid lines are parts of the boundaries of DN, HOV and EMP, from left to right.

We then consider the relation between $\mathcal{H}^{\mathbb{R}}$ and $\mathcal{H}^{\mathbb{C}}$. By the result of Bedford, Lyubich and Smillie, $\mathcal{H}^{\mathbb{R}} \subset \mathcal{H}^{\mathbb{C}} \cap \mathbb{R}^2$. It is then natural to ask what happens in $(\mathcal{H}^{\mathbb{C}} \cap \mathbb{R}^2) \setminus \mathcal{H}^{\mathbb{R}}$.

Definition 1 ([3]). We call $(a, c) \in \mathcal{H}^{\mathbb{C}} \cap \mathbb{R}^2$ is of type-1 if $(a, c) \in \mathcal{H}^{\mathbb{R}}$, and of type-2 if $K^{\mathbb{R}}_{a,c} = \emptyset$. Otherwise, it is of type-3.

Since $\mathsf{DN} \subset \mathcal{H}^{\mathbb{R}}$, the set of type-1 parameter values is non-empty. The set of type-2 parameter values is also non-empty since it contains $\mathsf{EMP} \cap \mathsf{HOV}$.

Conjecture 1 (Hubbard)**.** There exists a parameter value of type-3.

We prove that this conjecture is true.

Theorem 1 ([2])**.** *There exist parameter values of type-3.*

Besides the existence, Hubbard also conjectured that there are infinitely many classes of type-3 parameter values corresponding to mutually different real dynamics. This stronger conjecture is, to be precise, given in terms of the monodromy representation of the fundamental group of the hyperbolic horseshoe locus as below.

Denote by $\mathcal{H}_0^{\mathbb{C}}$ the component of $\mathcal{H}^{\mathbb{C}}$ that contains HOV. Let us fix a basepoint $(a_0, c_0) \in \mathsf{DN}$ and a topological conjugacy $h_0 : K^{\mathbb{C}}_{a_0,c_0} \to \Sigma_2$. Given a loop $\gamma : [0,1] \to \mathcal{H}_0^{\mathbb{C}}$ based at (a_0, c_0), we construct a continuous family of conjugacies $h_t : K^{\mathbb{C}}_{\gamma(t)} \to \Sigma_2$ along γ such that $h_0 = h_0$. Then we define

$$\rho(\gamma) := h_1 \circ (h_0)^{-1} : \Sigma_2 \to \Sigma_2.$$

It is easy to see that ρ defines a group homomorphism

$$\rho : \pi_1(\mathcal{H}_0^{\mathbb{C}}, (a_0, c_0)) \to \mathrm{Aut}(\Sigma_2)$$

where $\mathrm{Aut}(\Sigma_2)$ is the group of the automorphisms of Σ_2. Recall that an *automorphism* of Σ_2 is a homeomorphism of Σ_2 which commutes with the shift map σ. We call ρ *the monodromy homomorphism* and denote its image by Γ.

The monodromy homomorphism was originally defined for polynomial maps of one complex variable. In this one-dimensional case, it had been shown that the monodromy homomorphism is surjective regardless of the degree of the polynomial.Hubbard conjectured that the surjectivity also holds in the case of the complex Hénon map, with the only exception being σ.

Conjecture 2 (Hubbard)**.** The image Γ of the monodromy homomorphism and the shift map σ generate $\mathrm{Aut}(\Sigma_2)$.

The structure of $\mathrm{Aut}(\Sigma_2)$ is quite complicated and therefore, the conjecture implies, provided it is true, that the topological structure of $\mathcal{H}^{\mathbb{C}}$ is very rich.

Toward Conjecture 2, we obtain the following result.

Theorem 2 ([2])**.** *The order of the group Γ is infinite. In particular, it contains an element of infinite order.*

2. Monodromy and the Pruning Front

Apart form the theoretical interest, the monodromy theory of complex Hénon map can contribute to the understanding of the real Hénon map.

Let $(a, c) \in \mathcal{H}^{\mathbb{C}} \cap \mathbb{R}^2$. If (a, c) is of type-1 or 2, then by definition $K^{\mathbb{R}}_{a,c}$ is a full horseshoe, or empty. Suppose (a, c) is of type-3. We then ask what is $K^{\mathbb{R}}_{a,c}$ in this case. By definition, $K^{\mathbb{R}}_{a,c}$ is a proper subset of $K^{\mathbb{C}}_{a,c} \cong \Sigma_2$. The following theorem reveals that $K^{\mathbb{R}}_{a,c}$ is actually a subshift of Σ_2 which is realized as the fixed point set of the monodromy of a loop passing through (a, c).

Theorem 3. *For any $(a, c) \in \mathcal{H}_0^{\mathbb{C}} \cap \mathbb{R}^2$, there exists a loop $\gamma : [0,1] \to \mathcal{H}_0^{\mathbb{C}}$ with $\gamma(1/2) = (a, c)$ such that $H_{a,c} : K^{\mathbb{R}}_{a,c} \to K^{\mathbb{R}}_{a,c}$ is topologically conjugate to*

$$\sigma|_{\mathrm{Fix}(\rho(\gamma))} : \mathrm{Fix}(\rho(\gamma)) \to \mathrm{Fix}(\rho(\gamma)).$$

The conjugacy is given by the restriction of $h_{1/2}$ to $K^{\mathbb{R}}_{a,c}$.

As an application of Theorem 3, we can determine some subshifts of finite type that appear in the real Hénon map as follows. See §3 for the definition of I_p, I_q, I_r and I_s.

Theorem 4. *Let $(a, c) \in I_p$. Then the real Hénon map $H_{a,c} : K^{\mathbb{R}}_{a,c} \to K^{\mathbb{R}}_{a,c}$ is topologically conjugate to the subshift of Σ_2 with two forbidden blocks* 0010100 *and* 0011100*. Similarly, $K^{\mathbb{R}}_{a,c}$ is conjugate to the subshift of Σ_2 defined by the following forbidden blocks:*

$$\begin{array}{llll} 10100 & \textit{and} & 11100 & \textit{for } (a,c) \in I_q \\ 10010 & \textit{and} & 10110 & \textit{for } (a,c) \in I_r \\ 0010 & \textit{and} & 0110 & \textit{for } (a,c) \in I_s. \end{array}$$

Notice that I_p contains $(a, c) = (1, -5.4)$, the parameter studied by Davis, MacKay and Sannami [5]. The subshift for $(a, c) \in I_p$ given in Theorem 4 is equivalent to that observed by them. Thus, we can say that their observation is now rigorously verified.

Here we want to emphasise that this theorem is closely related to the so-called "pruning front" theory [4]. Theorem 3 implies that "primary pruned regions", or, "missing blocks" of $K^{\mathbb{R}}_{a,c}$ is nothing else but the region where the generating partition are interchanged along γ. In fact, notice that Figure 6 gives the complete description of primary pruned regions for the corresponding real Hénon map.

3. Computations and Proofs

Let us define subsets of the parameter space $\mathbb{C}^2$ by

$$\begin{aligned} L_p &:= \{a = 1, c \notin \text{white regions of Figure 1}\}, \\ L_q &:= \{a = 0.25, c \notin \text{white regions of Figure 2}\}, \\ L_r &:= \{a = -1, c \notin \text{white regions of Figure 3}\}, \\ L_s &:= \{a = -0.375, c \notin \text{white regions of Figure 4}\}. \end{aligned}$$

Let $L := L_p \cup L_q \cup L_r \cup L_s \subset \mathbb{C}^2$.

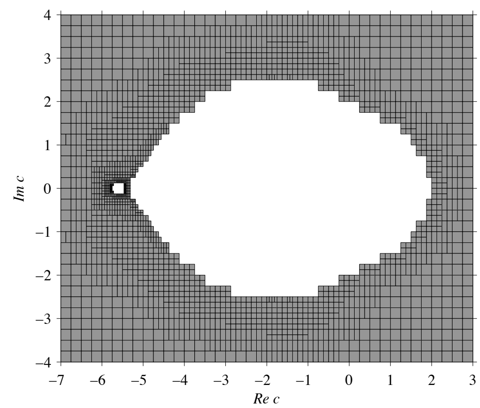


Figure 1: {The shaded region} $\subset \mathcal{H}^{\mathbb{C}} \cap \{a = 1\}$.

Lemma 5 ([2])**.** *If $(a, c) \in L$ then $H_{a,c}$ is uniformly hyperbolic on its chain recurrent set $\mathcal{R}(H_{a,c})$.*

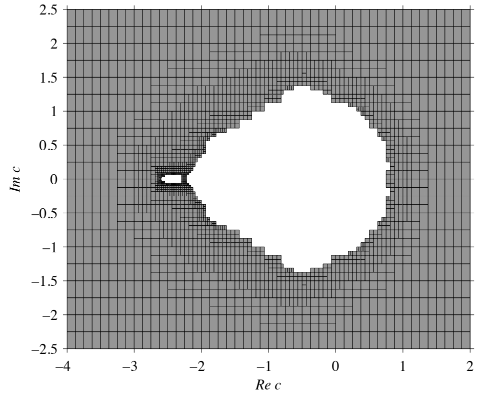


Figure 2: {The shaded region} $\subset \mathcal{H}^{\mathbb{C}} \cap \{a = 0.25\}$.

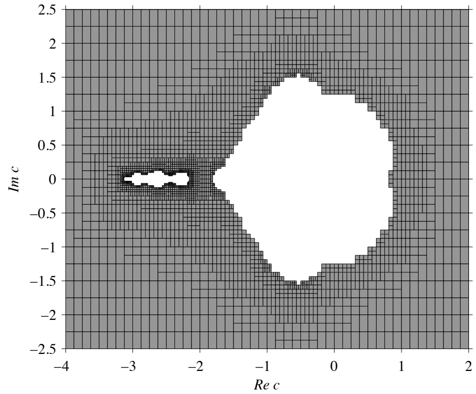


Figure 4: {The shaded region} $\subset \mathcal{H}^{\mathbb{C}} \cap \{a = -0.375\}$.

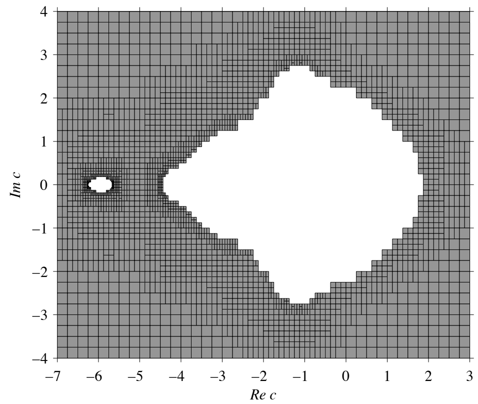


Figure 3: {The shaded region} $\subset \mathcal{H}^{\mathbb{C}} \cap \{a = -1\}$.

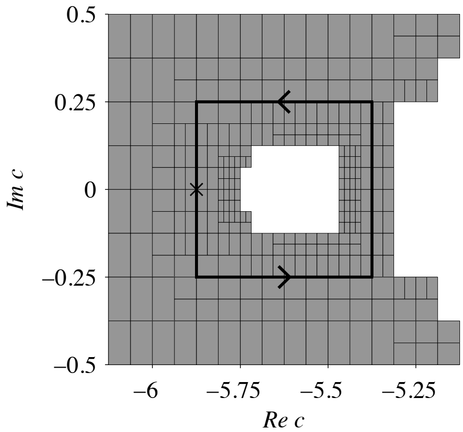


Figure 5: The loop β_p based at $(a, c) = (1, -5.875)$.

The proof of this lemma is computer assisted. The algorithm will be sketched briefly in §4.

Recall that the hyperbolicity of the chain recurrent set implies the $\mathcal{R}$-structural stability.Therefore, it follows from Lemma 5 that no bifurcation occurs in $\mathcal{R}(H_{a,c})$ as long as $(a, c) \in L$. Since L and DN have non-empty intersection and $K^{\mathbb{C}}_{a,c} = K^{\mathbb{R}}_{a,c} = \mathcal{R}(H_{a,c})$ is a hyperbolic full horseshoe on DN, we know that $\mathcal{R}(H_{a,c})$ is also a hyperbolic full horseshoe for all $(a, c) \in L$.

In general, $\mathcal{R}(H_{a,c})$ and $K^{\mathbb{C}}_{a,c}$ do not necessarily coincide. However, we can show that if $(a, c) \in L$ then $\mathcal{R}(H_{a,c}) = K^{\mathbb{C}}_{a,c}$ and therefore we can conclude $L \subset \mathcal{H}^{\mathbb{C}}$ (see [2]).

The set $L_p \cap \mathbb{R}^2$ have three components: two unbounded intervals, and one bounded interval connecting two white regions in Figure 1. We define I_p to be this bounded one. Similarly, I_q, I_r and I_s are defined to be the bounded intervals contained in $L_q \cap \mathbb{R}^2$, $L_q \cap \mathbb{R}^2$ and $L_s \cap \mathbb{R}^2$, respectively. We know that I_p, I_q, I_s and I_r are contained in $\mathcal{H}^{\mathbb{C}}_0 \cap \mathbb{R}^2$. To complete the proof of Theorem 1, we need to show that these intervals are of type-3.

For this purpose, we make use of Theorem 3. Since we have already shown that $L \subset \mathcal{H}^{\mathbb{C}}_0$, we can consider the monodromy of loops in L,

Let $\beta_p : [0, 1] \to L_p$ be a loop that turns around the smaller white island of Figure 1 as illustrated in Figure 5. We require that $\beta_p(1/2) \in I_p$, and that β_p be symmetric, that is, $\bar{\beta}_p = \beta_p^{-1}$. Then we define a loop $\gamma_p : [0, 1] \to L_p \cup \mathsf{HOV}$ based at $(1, -10) \in \mathsf{DN}$ by setting $\gamma_p := \bar{\alpha}^{-1} \cdot \beta_p \cdot \alpha$ where $\alpha : [0, 1] \to \mathsf{HOV} \cup L_p$ is a path from $(1, -10)$ to the basepoint of β_p. Choose the parametrization of γ_p so that $\gamma_p(1/2) \in I_p$ and $\bar{\gamma}_p = \gamma_p^{-1}$ hold. Similarly we define loops γ_q, γ_r and γ_s based at $(1, -10)$ turning around the smaller islands in L_q, L_r and L_s, respectively.

Proposition 6 ([2]). *The automorphism $\rho(\gamma_p)$ interchanges the words* 0010100 *and* 0011100 *contained in $s = (s_i)_{i \in \mathbb{Z}} \in \Sigma_2$. Namely,*

$$(\rho(\gamma_p)(s))_i = \begin{cases} 0 & \text{if } s_{i-3} \cdots s_i \cdots s_{i+3} = 0011100 \\ 1 & \text{if } s_{i-3} \cdots s_i \cdots s_{i+3} = 0010100 \\ s_i & \text{otherwise.} \end{cases}$$

Similarly, $\rho(\gamma_q)$ interchanges 10100 *and* 11100*, $\rho(\gamma_r)$ interchanges* 10010 *and* 10110*, and $\rho(\gamma_s)$ interchanges* 0010 *and* 0110*. See Figure 6.*

To prove the proposition, we follow the continuation of the symbolic partition of the horseshoe at the base point $(a, c) = (1, -10)$ using rigorous interval arithmetic. See [2] for the detail.

Now we are prepared to prove Theorem 1.

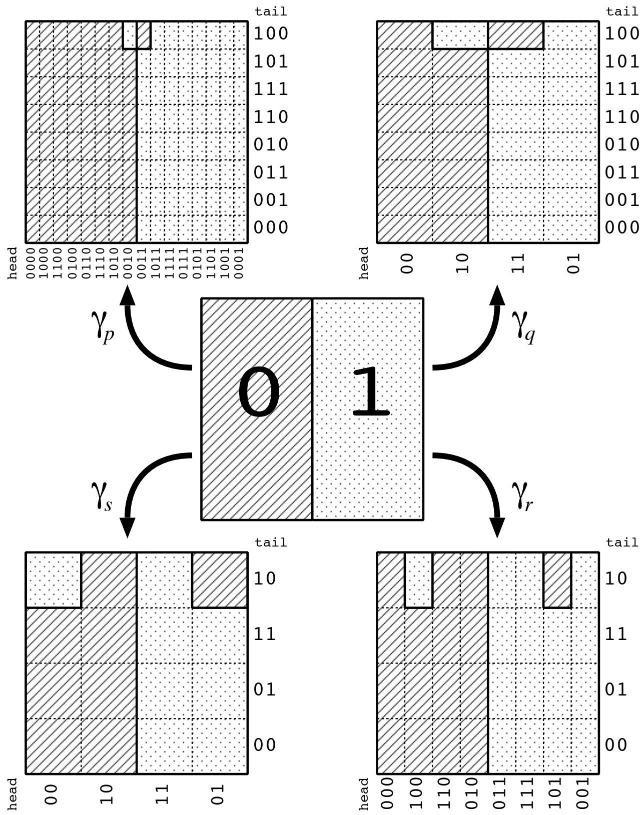


Figure 6: The change of the partition along loops.

Proof of Theorem 1. Since $\mathrm{Fix}(\rho(\gamma_p))$ is a non-empty proper subset of Σ_2, Theorem 3 implies that $\gamma_p(1/2) \in I_p$ is of type-3. By considering loops homotopic to γ_p, we can show that all $(a, c) \in I_p$ are also of type-3. □

Theorem 2 immediately follows from the following.

Proposition 7 ([2]). *Let $\gamma_\emptyset$ be a loop in $\mathcal{H}_0^{\mathbb{C}}$ based at (a_0, c_0) which is homotopic to the generator of $\pi_1(\mathsf{HOV})$. The the order of $\psi = \rho(\gamma_\emptyset) \cdot \rho(\gamma_s)$ is infinite.*

Theorem 4 is a direct consequence of Theorem 3 and Proposition 6.

The source codes of the programs for computer assisted proofs are available at the author's web site (`http://www.cris.hokudai.ac.jp/arai/`).

4. Algorithm for Proving Uniform Hyperbolicity

We recall an algorithm for proving the uniform hyperbolicity of chain recurrent sets developed by the author [1].

Let f be a diffeomorphism on a manifold M and Λ a compact invariant set of f. We denote by $T\Lambda$ the restriction of the tangent bundle TM to Λ.

In general, proving the uniform hyperbolicity following the conventional definition is quite hard. In particular, when the parameter is close to non-hyperbolic region, it is very difficult to construct a hyperbolic splitting.

To avoid this difficulty, we introduce the following notion. Consider $Tf : T\Lambda \to T\Lambda$ as a dynamical system. An orbit of Tf is said to be trivial if it is contained in the image of the zero section.

Definition 2. We say that f is *quasi-hyperbolic* on Λ if $Tf : T\Lambda \to T\Lambda$ has no non-trivial bounded orbit.

Uniform hyperbolicity implies quasi-hyperbolicity. The converse is not true in general. However, when $f|_\Lambda$ is chain recurrent, these two notions coincide.

Theorem 8 (Churchill-Selgrade, Sacker-Sell)**.** *Assume $\mathcal{R}(f|_\Lambda) = \Lambda$. Then f is uniformly hyperbolic on Λ if and only if f is quasi-hyperbolic on it.*

The definition of quasi-hyperbolicity can be rephrased in terms of isolating neighborhoods as follows. Recall that a compact set N is an isolating neighborhood with respect to f if the maximal invariant set $\mathrm{Inv}(N, f)$ is contained in $\mathrm{int}\, N$, the interior of N.

Proposition 9 ([1]). *Assume that N is an isolating neighborhood with respect to $Tf : T\Lambda \to T\Lambda$ containing the zero-section of $T\Lambda$. Then Λ is quasi-hyperbolic.*

In our case of the Hénon map, $\mathcal{R}(H_{a,c})$ is chain recurrent [2]. To prove Lemma 5, therefore, it suffices to show that $\mathcal{R}(H_{a,c})$ is quasi-hyperbolic for $(a, c) \in L$. By Proposition 9, all we have to do is to find an isolating neighbourhood containing the zero-section of $T\mathcal{R}(H_{a,c})$. More precisely, it is enough to find $N \subset TM$ such that

$$\mathcal{R}(H_{a,c}) \subset N \quad \text{and} \quad \mathrm{Inv}(N, TH_{a,c}) \subset \mathrm{int}\, N$$

hold. Here we identify $\mathcal{R}(H_{a,c})$ and its image by the zero-section. Since there are algorithms [1, Proposition 3.3] that efficiently compute rigorous outer approximations of $\mathcal{R}(H_{a,c})$ and $\mathrm{Inv}(N, TH_{a,c})$, these conditions can be checked on computers rigorously.

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