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Characterizing the phase synchronization transition of chaotic oscillators

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Abstract—The chaotic phase synchronization transition is studied in connection with the zero Lyapunov exponent. We propose a conjecture that it is associated with a switching of the maximal finite time zero Lyapunov exponent, which is introduced in the framework of large deviation analysis. A noisy sine circle map is investigated to introduce the conjecture and it is tested in an unidirectionally coupled Rössler system by using the covariant Lyapunov vector associated with the zero Lyapunov exponent.

1. Introduction

Many of cooperative behaviors in physical, chemical, and biological systems can be modeled by a coupled system of periodic oscillators. The important problem there is to clarify the condition under which the phases of periodic oscillators do synchronize. This phase synchronization problem has been extensively studied [1]. The oscillators are not always periodic but they are also chaotic in some practical cases. The phases of chaotic oscillators were also discovered to synchronize by Rosenblum et al. [2]. This chaotic phase synchronization (CPS) has been observed in experimental systems, e.g., electronic circuits [3], laser systems [4], and convective flows [5]. Thus CPS is also an important phase synchronization problem.

The study on synchronization of chaotic oscillators originated with the complete synchronization for coupled identical chaotic oscillators [6]. Other several types of chaotic synchronization including CPS have been also found [7]. Theoretical studies on chaotic synchronization have attempted to formulate its transition as a stability change of the system, and it was revealed that synchronization arises when one of Lyapunov exponents (LEs) switches its sign, for some types of chaotic synchronization [8, 9, 10]. CPS was also discussed in connection with the zero LE, which is one of LEs vanishing in the absence of coupling and is assumed to be connected with the dynamics of phase difference, but no obvious relation between them has been obtained. In fact, it was proved that no exact relation between the switching of the zero LE in its sign and the CPS transition exists [11, 12]. In this Letter, we show that the CPS transition is associated with a qualitative change not in the zero LE itself but in the fluctuation of zero LE. This is also a stability change of a new kind qualitatively different from those known for other types of chaotic synchronization.

2. CPS and discrete time dynamics

Let us consider the unidirectionally coupled Rössler oscillators

$$\begin{aligned} \dot{x}_d &= -\omega_d y_d - z_d, & \dot{x}_r &= -\omega_r y_r - z_r + \epsilon(x_d - x_r), \\ \dot{y}_d &= \omega_d x_d + a y_d, & \dot{y}_r &= \omega_r x_r + a y_r, \\ \dot{z}_d &= p + z_d(x_d - c), & \dot{z}_r &= p + z_r(x_r - c), \end{aligned} \tag{1}$$

where (x_d, y_d, z_d) and (x_r, y_r, z_r) are the coordinates of drive and response oscillators, respectively, and ϵ is the coupling strength. The parameter $\omega_{d(r)}$ controls the natural mean frequency of the drive (response) oscillator. In the following, we take $\omega_d = 0.93$ and $\omega_r = 0.95$, and the other control parameters are fixed at $a = 0.15$, $p = 0.2$, and $c = 10$. The system of Eq. (1) is known to show the CPS transition as ϵ is increased [11, 13]. The phase of drive (response) oscillator is introduced as the rotation angle $\phi_{d(r)}(t)$ around the origin in the $x_{d(r)}$-$y_{d(r)}$ plane and the phase difference $\theta(t)$ of the coupled system is defined as $\theta(t) = \phi_r(t) - \phi_d(t)$. In the presence of CPS, $\theta(t)$ is bounded and the mean phase velocity

$$\Omega \equiv \lim_{T\to\infty} \frac{\theta(T) - \theta(0)}{T} = \langle \dot{\theta}(t) \rangle \tag{2}$$

vanishes, where $\langle \cdot \rangle$ denotes the long time average.

Let us introduce the Poincare section defined by the condition $y_d = 0$ and $x_d > 0$, and t_n $(n = 1, 2, 3, \cdots)$ be the time of the nth crossing of the orbit, i.e., $\phi_d(t_n) = 2\pi n$. Then the CPS transition can be determined by the discrete time dynamics of the phase difference $\theta_n \equiv \theta(t_n) = \phi_r(t_n) - \phi_d(t_n)$. The relation between θ_n and θ_{n+k} $(k > 0)$ is approximately expressed as [14]

$$\theta_{n+k} \simeq \theta_n + f(\theta_n) + g(\theta_n)h(\mathbf{A}_n), \quad \langle h(\mathbf{A}_n) \rangle = 0, \tag{3}$$

where $f(\theta)$ and $g(\theta)$ are 2π-periodic functions, and $\mathbf{A}$ denotes the variables other than θ on the Poincare section. Let us denote the minimum and maximum values of $h(\mathbf{A}_n)$ by $h_{\min}$ and $h_{\max}$, respectively, and assume that $f(\theta)$ and $g(\theta)$ are continuous functions. Note that $h(\mathbf{A}_n)$ is always finite, since the attractor occupies a bounded region in the phase space. Then, a sufficient condition for the presence of CPS is expressed as follows: there is an interval I of θ such that, for any given $h \in [h_{\min}, h_{\max}]$, $\theta + f(\theta) + g(\theta)h$ is monotonically increasing in I and $f(\theta) + g(\theta)h$ has a pair of zeros in I [15]. Figure 1 shows the graph of θ_n-θ_{n+k} with $k = 15$ for Eq. (1) with $\epsilon = 0.033$ near the CPS transition point $\epsilon = \epsilon_c (\simeq 0.042)$, which is consistent with Eq. (3). The above condition of CPS will be applicable for phase coherent chaotic oscillators, but there are another possible mechanisms of CPS with different conditions from those for non-phase coherent ones [16, 20]. In the following, the phase coherence of chaotic oscillators is assumed.

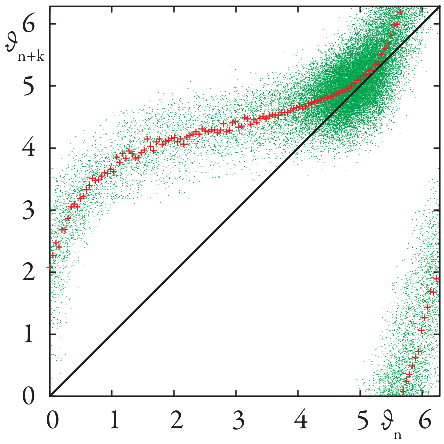

Figure 1: Graph of θ_{n+k} vs θ_n with $k = 15$ for Eq. (1) with $\epsilon = 0.033$. The symbol + shows the averaged result, i.e., $h(\mathbf{A_n})$ is averaged out.

As a stochastic model of Eq. (3), by replacing the chaotic modulation term $h(\mathbf{A}_n)$ with an independent random variable ξ_n, let us introduce the noisy sine-circle map [16]

$$\theta_{n+1} = \omega + \theta_n - \epsilon \sin \theta_n + \xi_n, \tag{4}$$

where $\omega > 0$ and $\epsilon \geq 0$ are control parameters corresponding to the average natural frequency difference and the coupling strength, respectively, and $g = 1$ and $k = 1$ is assumed for simplicity. Here we assume that ξ_n distributes uniformly over an interval $[-h_0, h_0]$ for simplicity. Note that, in this case, the mean phase velocity of Eq. (2) is replaced by $\Omega \equiv \lim_{N\to\infty}(\theta_N - \theta_0)/N$. The above condition of CPS for Eq. (4) holds if $\epsilon_c \equiv \omega + h_0 < \epsilon \leq 1$ is satisfied. It is also obvious that the phase desynchronization takes place for $\epsilon < \epsilon_c$, because the difference $\theta_{n+1} - \theta_n$ is not less than $\omega - \epsilon + \xi_n = \epsilon_c - \epsilon + \xi_n - h_0$ and it becomes positive with a finite probability. Thus the CPS transition takes place at $\epsilon = \epsilon_c$. In the following, we set $\omega = 0.4$ and $h_0 = 0.5$ that lead to $\epsilon_c = 0.9$.

Now let us investigate the zero LE in the noisy sine-circle map. For Eq. (4), there exists only one LE $\bar{\Lambda}$ that is calculated as

$$\bar{\Lambda} = \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} \lambda(\theta_k) = \langle \lambda(\theta_k) \rangle, \tag{5}$$

where $\lambda(\theta) \equiv \log|1 - \epsilon \cos\theta|$ is the local expansion rate. $\bar{\Lambda}$ is certainly the zero LE, which is assured by noting that $\bar{\Lambda} = 0$ at $\epsilon = 0$. It is easily evaluated that $\bar{\Lambda} \propto -\epsilon^2$ for Eq. (4). Furthermore, Fig. 3 shows the same dependence of $\bar{\Lambda}$ on ϵ for Eq. (1), which seems to be quite general behavior for the zero LE, as predicted by Ref. [12].

3. Finite time LE analysis

Instead of the zero LE itself, let us consider the finite time LE [17] defined as

$$\Lambda_n(\theta_0) = \frac{1}{n} \sum_{k=0}^{n-1} \lambda(\theta_k), \tag{6}$$

which converges to $\bar{\Lambda}$ in the limit $n \to \infty$. We here discuss the behavior of $\Lambda_n(\theta_k)$ for Eq. (4) in the present and absent cases of CPS. In the presence of CPS, i.e., $\epsilon > \epsilon_c$, all points of the orbit $\{\theta_n\}$ after initial transients are bounded as $\theta_- \leq \theta_n \leq \theta_+$, where $\theta_\pm$ satisfies $\omega \pm h_0 - \epsilon \sin(\theta_\pm) = 0$ and $0 < 1 - \epsilon \cos\theta_- < 1 - \epsilon \cos\theta_+ < 1$. Then the local expansion rate $\lambda(\theta_n)$ is always negative, and hence $\Lambda_n(\theta_k) < 0$ holds for any k. In the absence of CPS, i.e., $\epsilon < \epsilon_c$, the orbit $\{\theta_n\}$ can not be localized but it spreads out so that $\tilde{\theta}_n \equiv \theta_n \pmod{2\pi}$ distributes over the interval $[0, 2\pi)$. As a result, since $\lambda(\theta_n)$ becomes positive for $\pi/2 < \tilde{\theta}_n < 3\pi/2$, $\Lambda_n(\theta_k) > 0$ holds for some k. So we define $\Lambda_{\max}$ as

$$\Lambda_{\max} = \lim_{n\to\infty} \max_k \{\Lambda_n(\theta_k)\}. \tag{7}$$

The above discussion gives that $\Lambda_{\max} > 0 (< 0)$ holds in the absence (presence) of CPS.

In the case of $h_0 > \omega$, $\Lambda_{\max}$ can be determined. For $\epsilon > \epsilon_c$ Eq. (4) with $\xi_n = h_0$ has a stable fixed point at $\theta = \theta_+$. For $\epsilon < \epsilon_c$ Eq. (4) with $\xi_n = -\omega > -h_0$ has an unstable fixed point at $\theta = \pi$. At each of these fixed points $\lambda(\theta)$ takes its possible maximal value and, with a finite probability, there exists an arbitrarily long segment of the orbit keeping staying around it. Thus $\Lambda_{\max}$ is obtained as

$$\Lambda_{\max} = \begin{cases} \log(1 + \epsilon), & \epsilon < \epsilon_c, \\ \log\left(1 - \sqrt{\epsilon^2 - \epsilon_c^2}\right), & \epsilon > \epsilon_c \end{cases} \tag{8}$$

showing a discontinuity at the CPS transition point $\epsilon = \epsilon_c$. Note that $\Lambda_{\max}$ converges to zero obeying a scaling law $\Lambda_{\max} \propto -\sqrt{\epsilon - \epsilon_c}$ for $\epsilon > \epsilon_c$. In summary, the CPS transition is characterized by $\Lambda_{\max}$ switching discontinuously between a positive and zero.

In Fig. 2, Eq. (8) is compared to the numerically evaluated $\Lambda_{\max}$ with two different ensemble numbers N. Since, in the present case, the possible maximal value of $\lambda(\theta_k) = \Lambda_1(\theta_k)$ coincides with $\Lambda_{\max}$ as used in the derivation of Eq. (8), $n = 1$ is used in order to save the ensemble number N. Figure 2 suggests that the numerical result will converge to the theoretical one as $N \to \infty$.

Here we propose a conjecture that $\Lambda_{\max}$ of the zero LE characterizes the CPS transition, i.e., it switches discontinuously between a positive and zero at the transition to CPS,

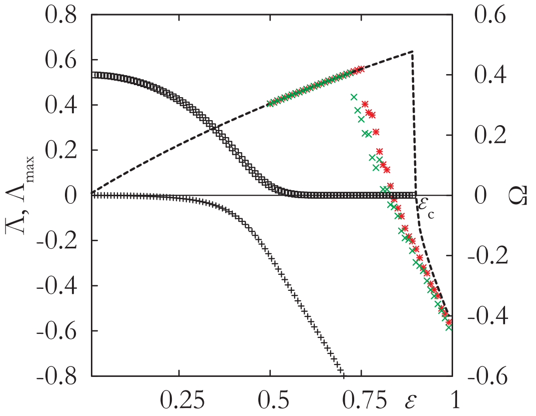

Figure 2: $\Lambda_{\max}$ vs ϵ for Eq. (4) with $n = 1$. Numerical results with the ensemble numbers $N = 10^8(\times)$ and $10^{10}(*)$ are compared with the theoretical one in Eq. (8) (dashed line). The zero LE $\bar{\Lambda} = \Lambda(0)(+)$ and the mean phase velocity $\Omega(\square)$ numerically evaluated with $N = 10^8$ are also plotted.

in general systems as that of Eq. (1). Let us numerically check the validity of this conjecture for the unidirectionally coupled Rössler system of Eq. (1). We first introduce a continuous time system expressed as $\dot{\mathbf{X}} = \mathbf{F}(\mathbf{X})$ with the Jacobi matrix $\hat{G}(\mathbf{X})$ of $\mathbf{F}(\mathbf{X})$ at $\mathbf{X}$. For each of LEs, there is an associated tangent vector called covariant Lyapunov vector (CLV) $\mathbf{v}(t)$ along the orbit $\mathbf{X}(t)$ on the attractor satisfying $\dot{\mathbf{v}}(t) = \hat{G}(\mathbf{X}(t))\mathbf{v}(t)$. The corresponding LE is given by $\lim_{t\to\infty} \frac{1}{t} \log \frac{|\mathbf{v}(t)|}{|\mathbf{v}(0)|} = \lim_{t\to\infty} \frac{1}{t} \int_0^t \frac{\mathbf{v}(s)\cdot\hat{G}(\mathbf{X}(s))\mathbf{v}(s)}{\mathbf{v}(s)\cdot\mathbf{v}(s)} ds$. For each CLV, the finite time LE can be defined as

$$\Lambda_t(\mathbf{X}(0), \mathbf{v}(0)) \equiv \frac{1}{t} \int_0^t \lambda(\mathbf{X}(s), \mathbf{v}(s)) ds \tag{9}$$

with the local expansion rate

$$\lambda(\mathbf{X}(s), \mathbf{v}(s)) \equiv \frac{\mathbf{v}(s) \cdot \hat{G}(\mathbf{X}(s))\mathbf{v}(s)}{\mathbf{v}(s) \cdot \mathbf{v}(s)}. \tag{10}$$

In the following, we take the CLV associated with the zero LE as $\mathbf{v}(t)$.

Now we apply the analysis to the system of Eq. (1), whose zero LE is specified as follows: Among six LEs in Eq. (1), three of them are those of the drive oscillator alone and they are excluded. The zero LE is the second largest one of the others. Indeed it vanishes at $\epsilon = 0$ and becomes negative for $\epsilon > 0$, as shown in Fig. 3.

Equation (1) is numerically integrated in terms of the 4th-order Runge Kutta method over a given time interval to construct a mapping system, and then the QR decomposition method [18] and the method by Ginelli et al. [19] are used to obtain the zero LE $\bar{\Lambda}$ and the CLV $\mathbf{v}(t)$ associated with $\bar{\Lambda}$, respectively. Figure 3 shows $\Lambda_{\max}$ and Ω as a function of ϵ, where $t = 3000$ and $T = 1.8 \times 10^8$ are used. It is well known that Ω, in the vicinity of the onset of CPS, obeys an anomalous scaling law $\log\Omega \propto -(\epsilon_c - \epsilon)^{-1/2}$ for $\epsilon < \epsilon_c$ [7]. From the plot of $1/\log\Omega$ in Fig. 3, it is estimated that $\epsilon_c \simeq 0.042$, while $\Lambda_{\max}$ jumps at $\epsilon \simeq 0.04$ that nearly equals to ϵ_c. Furthermore, in the vicinity of the onset of CPS, it is observed that $\Lambda_{\max}$ obeys a scaling law $\Lambda_{\max} \propto -\sqrt{\epsilon - \epsilon_c}$ for $\epsilon > \epsilon_c$ and $\Lambda_{\max} \simeq$ Const. for $\epsilon < \epsilon_c$, which are consistent with those in Eq. (8) for the noisy sine-circle map. As a result, it is confirmed that $\Lambda_{\max}$ switches its sign discontinuously at the onset of CPS .

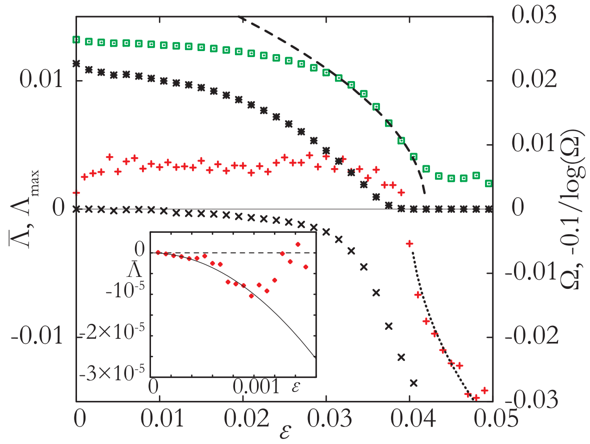

Figure 3: ϵ dependence of $\Lambda_{\max}(+)$, $\bar{\Lambda}(\times)$, and $\Omega(*)$. $-1/\log\Omega(\square)$ is also plotted to estimate the transition point $\epsilon = \epsilon_c$ with a fitting curve of $-1/\log\Omega \propto \sqrt{\epsilon_c - \epsilon}$ (dashed line), where it saturates for large ϵ due to finite T in numerically evaluating Eq. (2). Dotted line shows a curve of $-\sqrt{\epsilon - 0.04}$. The inset shows ϵ dependence of $\bar{\Lambda}$ for $0 < \epsilon \ll \epsilon_c$ and a solid line of $-\epsilon^2$.

Finally let us discuss our result in connection with unstable periodic orbits (UPOs) within the attractor. Pikovsky et al. have formulated the CPS transition for periodically driven chaotic oscillators as the phase locking of all UPOs embedded in the attractor to the external periodic driving [21]. In a similar manner, the CPS transition in the case of coupled system of two phase coherent chaotic oscillators would be formulated as follows: almost all the pairs of UPOs among different oscillators in the absence of coupling correspond to the unstable phase unlocked quasiperiodic orbits embedded in the attractor of the whole system, each of which bifurcates into a pair of UPOs by a saddle-node bifurcation as increasing the coupling strength and then the disappearance of the last unstable quasiperiodic orbit within the attractor leads to the CPS transition. To be concrete, let us consider the system of Eq. (1). The unstable quasiperiodic orbits take the signs of Lyapunov spectrum as $(+, +, 0, 0, -, -)$, each of which bifurcates into a pair of UPOs with those as $(+, +, 0, -, -, -)$ and $(+, +, +, 0, -, -)$, i.e., UPOs with the zero LEs of opposite signs. If the CPS is achieved, the attractor shrinks to exclude the UPOs with positive zero LEs from its inside. Thus the attractor contains UPOs with only negative zero LEs in the presence of CPS, while it contains UPOs with both positive and negative zero LEs in the absence of CPS.

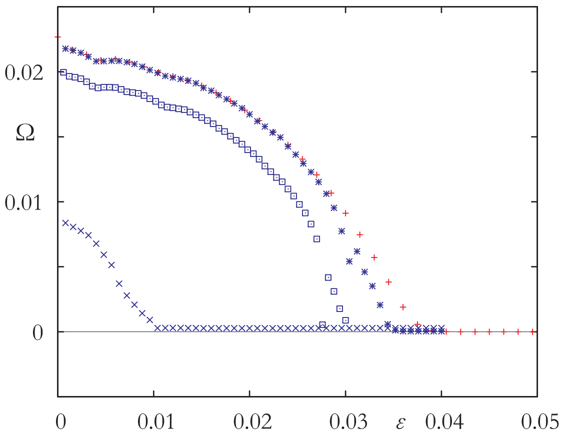

Figure 4: Ω vs ϵ for the systems driven by chaos (+) and UPOs of 1-period (×), 3-period (∗), and 5-period (□).

In order to assure the above picture, drive system is replaced by a UPO embedded in the attractor of the drive oscillator, and Ω is evaluated. In Fig. 4, the result for UPOs of 1-, 3-, and 5-period are shown, and they are comapred with Ω of the original system. It shows that each Ω driven by the UPO becomes 0 for $\epsilon < \epsilon_c$. The result indicates that all of UPOs embedded in the attractor of the response system are phase locked with a UPO of the drive system with 1-, 3-, and 5-period before the last UPO pairs in the response and drive system being phase locked. So the UPO in the drive system determining the CPS is something excluding the UPO obtained here.

4. SUMMARY

In conclusion, in contrast to the other types of chaotic synchronization, the CPS transition is characterized not by the LE itself but by a qualitative change in the fluctuation of LE. Namely, we found a relation between the maximal finite time zero LE and the CPS transition. $\Lambda_{\max}$ shows no critical behavior for $\epsilon < \epsilon_c$, but it discontinuously switches from a positive to zero at the CPS transition point $\epsilon = \epsilon_c$ and obeys a critical scaling law $\Lambda_{\max} \propto -\sqrt{\epsilon - \epsilon_c}$ for $\epsilon > \epsilon_c$.

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