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Social Network Formation Model Considering Penalties against Social Rationality Reduction

Tetsuo Imai

Graduate School of Information Sciences, Hiroshima City University
3–4–1 Ozuka-Higashi, Asaminami-ku, Hiroshima, 731–3194, Japan
e-mail: imai@hiroshima-cu.ac.jp

Abstract— Author and co-researcher have proposed the dynamic network formation game model for modeling complex networks. This model is based on the network formation game which is known in the field of game theory and it represents dynamical network formation process derived by many distributed decision making by self-interest agents. In this model, varied players' payoff function leads players' strategy to change, thus different outcome networks are generated. Therefore if the payoff function can be controlled, the structure of the network formed by a large number of players can be controlled in some degree. In the event of pandemics of infectious diseases such as COVID-19, it is important to change the structure of the social and infection transmission network. In this article, I examine a method for balancing individual and social rationality in social network formation using the dynamic network formation game model. A payoff function reflecting individual and social rationality is introduced, and the characteristics of generated network structures are investigated through computer simulation. The results showed that the resulting networks tended to be unnatural networks as social networks, but by introduced payoff function, structures with high resistance to the spread of infectious diseases were obtained.

1. Introduction

Author and co-researcher have proposed the dynamic network formation game model for modeling complex networks [1]. This model is based on the network formation game which is known in the field of game theory and it represents dynamical network formation process derived by many distributed decision making by self-interest agents. Authors have already investigated what types of network are generated by this model through computer simulations, and shown that this model can generate the scale-free networks in certain conditions, which is often observed in social and technological networks. In this model, varied players' payoff function leads players' strategy to change, thus different outcome networks are generated. Therefore if the payoff function can be controlled, the structure of the network formed by a large number of players can be controlled in some degree.

ORCID iDs Tetsuo Imai: 0000-0002-7202-0495

Social networks are not only information exchange networks among people, but also infectious disease transmission networks. Many studies on mathematical models of infectious disease transmission have shown that the structure of the network has a direct impact on transmission efficiency. Therefore, in order to control the spread of infectious diseases during the epidemic period, it is important to change the structure of the infection transmission network (i.e, social network) by some means.

In the case of COVID-19, which has been spreading around the world since the beginning of 2020, governments have attempted to control its spread by encouraging people to change their behavior patterns and changing the structure of social networks through various programs, such as requesting people to refrain from unnecessary and hasty travel and to refrain from opening restaurants at night. However, as strong infection control policies began to have a significant negative impact on social and economic activities, and as it became clear that suppression of COVID-19 was difficult to achieve in a short term, it became necessary not only to control the spread of infection, but also to promote social and economic activities to a certain degree.

For the promotion of social and economic activities, it is important to construct information exchange networks among people based on individual rationality. For example, in researches on the relationship between innovation and social network structure, the role of nodes and links that fill the gaps between local and dense partial networks (known as *structural holes*) is considered important. Thus, in the context of individual rationality, social members have strong incentives to form such links. However, since social links for person-to-person information exchange often involve contact, the existence of nodes and links that fill these structural holes can also dramatically facilitate the spread of infectious diseases. Therefore, in the context of social rationality, the formation of such links should be restricted. The optimization of social network structure basically faces such a dilemma. On the other hand, the appropriate social network structure differs depending on the stage of spread of the infectious disease. That is, strong control of the spread should be prioritized during the expansion phase, while recovery of social and economic activities should be prioritized during the contraction phase. Therefore, according to the target social network structure to be derived, it is necessary to have an optimization method for the social network structure that can control the balance between the intensity of individual and social rationality. Furthermore, in a liberal society, the formation of social networks is primarily based on the free will of individuals, so such a balance control method should be implemented as policies that reflect the optimality of social rationality in individual decision making.

In this article, I examine a method for balancing individual and social rationality in social network formation using the dynamic network formation game model. First, a payoff function reflecting individual and social rationality is introduced. Next, the characteristics of structures of the network generated by this model are investigated through computer simulations.

2. Methods

2.1. Dynamic Network Formation Game Model

A model of network formation used in this article, the dynamic network formation game model is described as follows. In this model, at each discrete time t, undirected and unweighted network $g(t)$ define a strategic form game of n players. Here n is the number of nodes of the network g and invariant with respect to time. The set of strategies of the n players game determines the network $g(t+1)$ at the next timestep,

As for the game which played at each timestep, *players* are nodes who intend to improve their payoffs of next timestep. The *strategy* of each player i is described as a vector $\boldsymbol{s}_i(t) = (s_{i1}(t), \dots, s_{i(i-1)}(t), s_{i(i+1)}(t), s_{in}(t))$, where $s_{ij}(t) \in \{0, 1\}$ represents a link formation request of i to j. The player i independently sets $s_{ij}(t)$ according to change of its payoff value with change of link ij, $s_{ij}(t)$ is set to 1 if it is desirable for i to add (or maintain) the link with player j, otherwise $s_{ij}(t)$ is set to 0. The payoff function $u(g)$ is a function that give a value for each player, indicating how the network $g(t)$ is desirable for the player. This function has a very strong influence on the strategies of the players and on the outcome solution network. The specifics are described in the subsection 2.2. As for the *outcome* $g(t+1)$ obtained by playing the game at time t, I describe about two concepts *adjacent* and *defeat*. Two networks g and g' are *adjacent* if the g' differs only one link from g, and a network g' *defeats* an adjacent network g if either

$$g' = g - ij \text{ and } (u_i(g') > u_i(g) \text{ or } u_j(g') > u_j(g))$$

or

$$\begin{aligned} g' = g + ij \text{ and } \\ \{(u_i(g') \geq u_i(g) \text{ and } u_j(g') \geq u_j(g)), \\ \text{except } (u_i(g') = u_i(g) \text{ and } u_j(g') = u_j(g))\} \end{aligned}$$

where $g + ij$ is a network which a link ij is added to network g, $g - ij$ is a network which a link ij is removed from network g. That is, g' is obtained by adding a link ij to network g that increases the payoffs of both players i and j, or by removing a link ij that increases the payoffs of either player i or j.

The $g(t+1)$ is specified deterministically among networks which can defeats $g(t)$ and $g(t)$ itself. For concrete description of $g(t+1)$, two definitions $\Delta u_{ij}(t+1)$ and *acceptable link set $L_{acceptable}(g)$* are describe as follows. First, $\Delta u_{ij}(t+1)$ is defined as the amount of change of i's payoff in the case that the change of link ij occurs at time step t. It is formulated as follows,

$$\Delta u_{ij}(t+1) = \begin{cases} u_i(g(t) + ij) - u_i(g(t)), & \text{if } \; ij \notin g(t) \\ u_i(g(t) - ij) - u_i(g(t)), & \text{if } \; ij \in g(t). \end{cases}$$

Second, $L_{acceptable}(g) \subset L \subset N \times N$ is defined as the set of links which are acceptable for involved players i and j. It is formally described as follows.

$$\begin{aligned} L_{acceptable}(g) = \{ij | ij \notin g \text{ and } g + ij \textit{ defeats } g\} \\ \cup \{ij | ij \in g \text{ and } g - ij \textit{ defeats } g\}. \end{aligned}$$

The link ij that changes at timestep t is described as the link that improves payoff the most among $L_{acceptable}(g)$. In formally,

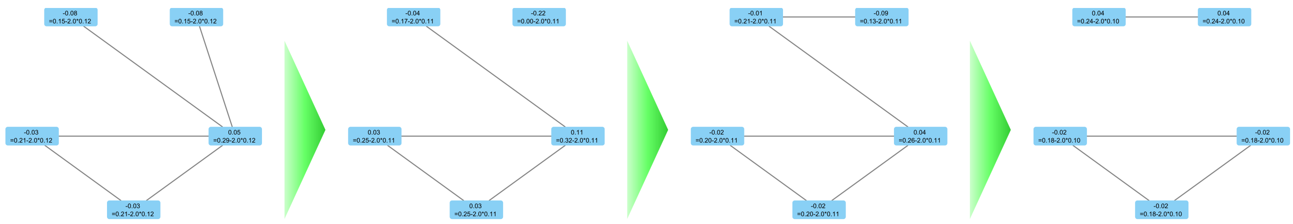
$$ij = \underset{ij \in L_{acceptable}(g(t))}{\arg\max} \; \Delta u_{ij}(t+1).$$

The outcome network $g(t+1)$ is determined as $g(t) + ij$ (if $g(t) \notin ij$) or $g(t) - ij$ (if $g(t) \in ij$). If there is more than one link satisfying that, the link which involved by the node who have the youngest ID is prior than others as a matter of convenience.

Every process of network formation starts from the initial state of network $g(0)$ and continues link changes. Following the representation manner of the dynamical system, I consider the network $g(t)$ at each time t as a *state* and the process by this model as a state-transition process. If the process reach to the state $g(t)$ in which no links are acceptable, then $g(t+1)$ is exactly same as $g(t)$, and the state $g(t)$ is described as *pairwise stable* which is one of solution of the process. Another solution is described as *improving cycle*, which consisted by a sequence of adjacent states $\{g_1, g_2, \dots, g_K\}$ such that each defeats the previous one and $g_1 = g_K$.

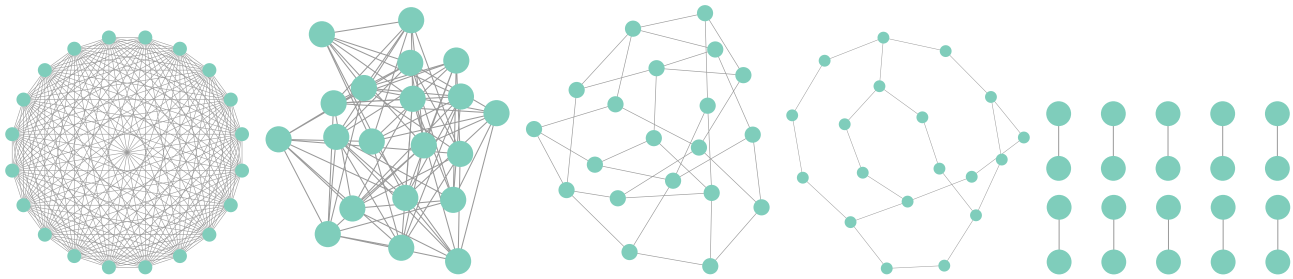
2.2. Payoff function

I describe here a payoff function that reflects individual and social rationality. In this paper, individual rationality maximization is represented as maximization of *information centrality*, and social rationality maximization is represented as minimization of *the largest eigenvalue of the adjacency matrix*. Information centrality is one measure of the centrality of nodes in a network. It was first proposed by Stephenson and Zelen as a measure to overcome the drawbacks of betweenness centrality, namely that



(a) $t = 0$ (b) $t = 1$ (c) $t = 2$ (d) $t = 3$(stable)

Figure 1: An example of process of the dynamic network formation game model of which parameters are $n = 5, m = 5, w = 2$. (a) Initial state $g(0)$. (b) A link is removed because of payoff of right node increase. (c) A new link is added because payoffs of the involved nodes are both increased. (d) Another link is removed. This network is *pairwise stable* and this state is a solution of the process.



(a) $w = 0$ (b) $w = 10$ (c) $w = 30$ (d) $w = 50$ (e) $w = 100$

Figure 2: Typical solutions of the model for each w. (a) Complete network which all node pair have links between them. (b) All node have 8 links. (c) Nodes have 3 or 4 links. (d) Nodes have 2 or 3 links. (e) Every node pair constitutes a component of size 2 each.

only the shortest pathes are used for information propagation and that the values of propagating information are not decayed depending on the distance between nodes [3]. It was later redefined by Brandes and Fleischer as current-flow closeness centrality, and a fast calculation algorithm was proposed [2]. Information centrality is basically similar to betweenness centrality, where nodes that are on more information transfer paths are evaluated more highly. On the other hand, information centrality differs from betweenness in that it considers all paths between node pairs and is weighted by the reciprocal of the path length so that shorter paths are evaluated more highly. Therefore, this is also similar to closeness centrality. In this article, I adopt this information centrality as a value reflecting individual rationality. That is, each player decides his/her strategy so that his/her information centrality is high in the outcome network of next timestep. Note that I use the value of relative information centrality (hereafter referred to as RIC), which is normalized so that the sum of the information centrality of all nodes is 1.

In propagation and diffusion over a network, it is known that the larger the maximum eigenvalue $\lambda_{\max}$ of the adjacency matrix of the network, the more efficient the diffusion [4]. Therefore, considering that I want to obtain networks to suppress the spread of infectious diseases, I adopt this value as a value that reflects social rationality. That is, each player decides his/her strategy taking into account that the maximum eigenvalue of the adjacency matrix will not be large in the outcome network. Specifically, the payoff function $u_i(g)$ for network g is represented as follows,

$$u_i(g) = RIC_i - w\lambda_{\max}. \tag{1}$$

Here w is a balancing parameter between individual and social rationality. If w is 0, players will pursue only individual rationality maximization, while a large value will cause players to adopt strategies that place more emphasis on social rationality. Figure 1 shows an example of the transition process of the network formation game model.

3. Simulations and results

For every simulations, the initial network $g(0)$ was randomly generated by the $G(n, m)$ model (called also as generalized random graph model). This model generates a network by choosing uniformly at random from the collection of all networks which have n nodes and m edges, here $n = 20, m = 38$ (link existence ratio 0.2). The weight w in equation (1) is set to one of $0, 10, 20, \ldots, 100$.

As a result, single-state solutions, i.e., pairwise stable networks, were obtained in all simulations. Figure 2 shows a sample of typical solutions for each of the weights $w = 0, 10, 30, 50$, and 100. When $w = 0$, only the information centrality matters completely, so there is a strong tendency

Table 1: Distribution of average degrees of the outcome network for each w. This result was obtained for 50 random initial networks.

Ave. degree

w	[0.0, 0.5)	[0.5, 1.0)	[1.0, 1.5)	[1.5, 2.0)	[2.0, 2.5)	[2.5, 3.0)	[3.0, 3.5)	[3.5, 4.0)	[4.0, 4.5)	[4.5, 5.0)	[5.0, 5.5)	[5.5, 6.0)	[6.0, 6.5)	[6.5, 7.0)	[7.0, 7.5)	[7.5, 8.0)	[8.0, 8.5)	[8.5, 9.0)	[9.0, 9.5)	[9.5, 10.0)
0	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
10	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.02	0.02	0.00	0.06	0.00	0.00	0.00	0.00	0.80	0.06	0.04	0.00
20	0.00	0.00	0.00	0.04	0.06	0.00	0.00	0.00	0.06	0.02	0.82	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
30	0.00	0.00	0.08	0.02	0.00	0.00	0.86	0.04	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
40	0.00	0.00	0.10	0.00	0.00	0.12	0.78	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
50	0.00	0.00	0.26	0.02	0.50	0.20	0.02	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
60	0.00	0.00	0.68	0.04	0.28	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
70	0.00	0.00	0.92	0.04	0.04	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
80	0.00	0.00	0.86	0.08	0.06	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
90	0.00	0.00	0.90	0.06	0.04	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
100	0.00	0.00	0.84	0.14	0.02	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00

w	[10.0, 10.5)	[10.5, 11.0)	[11.0, 11.5)	[11.5, 12.0)	[12.0, 12.5)	[12.5, 13.0)	[13.0, 13.5)	[13.5, 14.0)	[14.0, 14.5)	[14.5, 15.0)	[15.0, 15.5)	[15.5, 16.0)	[16.0, 16.5)	[16.5, 17.0)	[17.0, 17.5)	[17.5, 18.0)	[18.0, 18.5)	[18.5, 19.0]
0	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.02	0.00	0.00	0.08	0.00	0.00	0.00	0.00	0.00	0.00	0.90
10	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
20	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
30	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
40	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
50	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
60	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
70	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
80	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
90	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
100	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00

to form complete networks in which links exist between all nodes. As the value of w increases, the average degree tends to decrease. In all cases, the degree of each node was almost the same or similar. In the case of $w = 100$, many of the nodes were split into connected components of size 2 in many cases. Table 1 shows the distribution of average degrees in the outcome network for the 50 random initial networks. The rows show each value of w, the columns show each average degree, and each cell shows the proportion of occurrences. For example, for outcome networks with $w = 0$, the proportion of occurrences with a average degree of $[18.5, 19.0]$ was 0.9, for $[13.5, 14.0)$ it was 0.02, and for $[15.0, 15.5)$ it was 0.08. As shown also in the table, the average degree in the generated network tends to decrease as the value of w increases, and for $w = 70$ and above, the network with an average degree of 1 is pairwise stable in many cases.

Real-world social networks are considered to be scale-free and small-world. Scale-free property means that there is a large gap in the degree of each node and the degree distribution follows a power law, while small-world property means that localized, densely connected small groups are interconnected by sparse links. On the other hand, the outcome networks obtained from the simulations often have almost constant values for node degrees. The possible causes of this difference are as follows. When a node has a large degree, the maximum eigenvalue of the adjacency matrix of the entire network becomes large, allowing other players to form the same number of links as the node with the largest degree to increase their own information centrality without having a negative impact. Therefore, such links are more likely to be formed. This is the effect of adding a penalty term representing social rationality to equation (1), and in fact resulting networks have low maximum eigenvalues, however they tend to be unnatural networks as social networks. Similarly, clustering properties tended to be very small than actual social networks. This is due to the use of information centrality for individual rationality, which assigns low values to links with nodes that are already close to each other.

As a future issue, it is necessary to brush up the model so that it can represent more natural social network formation, and it is also necessary to examine the appropriate setting of the payoff function. It is also necessary to examine an appropriate method for determining w, a parameter that balances individual and social rationality. Furthermore, it is assumed that each player can observe the entire social network, but in reality, players often only know the situation in their own neighborhood, so it may be necessary to modify their strategy decisions. Although there are still many issues to be addressed, the method of inducing social networks formed based on selfish and decentralized decision making to desiable structures by setting an appropriate payoff function is expected to make a significant contribution to policy making, especially during the spread of infectious diseases. I hope that further studies will be conducted.

Acknowledgements

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