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Criterion for Judging Bifurcation of High-Dimensional Torus

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Abstract—
In this report, we explain a criterion for judging bifurcation of quasi-periodic solutions which are high-dimensional torus. We use two concepts called Dominant Lyapunov Exponent (DLE) and the corresponding Dominant Lyapunov Bundle (DLB) for the criterion.

1. Introduction

A quasi-periodic solution composed of more than two asynchronous frequency components, covers more than two-dimensional torus densely [1] (p. 189), [2], [3] (p. 16). Different from periodic solutions, we cannot apply Newton's method for obtaining a saddle quasi-periodic solution [4], because it cannot be a fixed point (including a periodic solution). Recently, we develop an algorithm for obtaining a saddle quasi-periodic solution sandwiched by two different attractors by using bisection method [5]. Via this algorithm, we are succeeded in observing a saddle-node bifurcation such that a stable quasi-periodic solution and a saddle quasi-periodic solution merge together and disappear. To analyze the bifurcation type of quasi-periodic solution, we define Lyapunov Bundle (LB) composed of many Lyapunov Vectors (LV) defining stable or unstable direction of Lyapunov Exponents (LEs) [6], [7], [8]. In continuous-time dynamical systems, LV covers the solution orbit densely, therefore the number of LBs equals to system dimension.

2. Dominant Lyapunov exponent and dominant Lyapunov bundle

The most important LB for local bifurcation is Dominant Lyapunov Bundle (DLB) corresponding to Dominant Lyapunov Exponent (DLE). We can judge the type of bifurcation by analyzing the type of DLB. The DLE is the non-zero LE which is closest to zero. We can know the bifurcation point from the point at which DLE touches zero in Lyapunov diagram and also bifurcation type from the DLB just before the bifurcation point.

In practice, when quasi-periodic solution is 2-torus in continuous-time dynamical system (flow), we can judge the bifurcation type as follows: 1. Compute LEs and LBs in parallel to the original 2-torus in flow (FT2). 2. Apply an appropriate Poincare section to FT2 and its associated LBs, then we can get 1-torus in discrete-time dynamical system (map) together with its LBs. 3. Judge the DLB type and classify the bifurcation type.

But, when a quasi-periodic solution is 3-torus or higher, if we apply Poincare section to 3-torus in flow (FT3) and its associated LBs, then we get DLB of 2-torus in map (MT2). But, it is still difficult to judge the DLB type, because DLB of MT2 is on 2-dimensional surface. Accordingly, we take a codim 2 section to MT2 and its associated LBs to get 1-torus in section (ST1) together with its associated LBs. By using this LBs of ST1, we can judge the bifurcation type of FT3.

The DLB types of FT3 are classified in 3 types; A^3, M^3 and F^3. To analyze these types, we apply Poincare section (codim 1 section) and a codim 2 section to FT3. The codim 2 section obtained from Poincare mapped MT2 enables us to get a ST1 as shown in Fig. 1. In numerical calculation, we take thin slice around the codim 2 section from MT2 to obtain ST1. The codim 2 section does not inherit dynam-

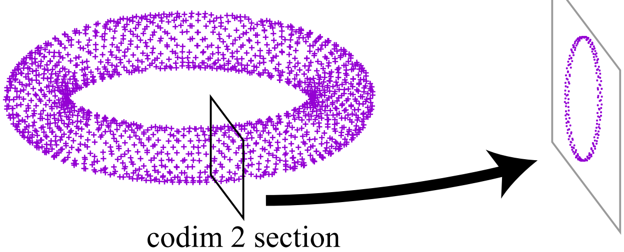


Figure 1: Schematic of a codim 2 section of MT2 and the corresponding ST1.

ics of the original system different from Poincare section. Therefore, in the codim 2 section, the A^- type DLB does not exist, but the both-side DLB which has annulus shape can exists. We call this DLB the Annulus Star (A^*) type DLB. The A^* type DLB causes Component Doubling (CD) bifurcation. The CD bifurcation of ST1 is equivalent to the DC bifurcation of FT3 occurring on the codim 2 section. When we take the CD bifurcation into account, the DLB types of FT3 are classified in 3 types; $A \times A \times A$,

$M \times A^* \times A^*$ and $F \times F \times F$. Therefore, we compute LB of MT2 obtained from FT3 and extract DLB of ST1 on a certain codim 2 section. If the type of DLB of ST1 is of the A^+ type, the original FT3's DLB type is of the A^3 which causes saddle-node bifurcation of FT3. If the type of DLB of ST1 is of the M or A^* type, the original FT3's DLB type is of the M^3 causing double covering bifurcation of FT3. If the type of DLB of ST1 is of the F type, the original FT3's DLB type is of the F^3 causing Neimark–Sacker bifurcation of FT3.

3. Example of Bifurcation of FT3

As an example of the bifurcation of FT3, we take Phase-Locked Loop (PLL) circuit with three periodic inputs. The normalized circuit equation of the PLL circuit is as follows:

$$\left\{\begin{array}{rcl} \dot{x}_0 & = & x_1, \\ \dot{x}_1 & = & -\beta x_1 - \sin x_0 + \beta\sigma + m_1\beta \sin x_2 + m_1\omega_1 \cos x_2 \\ & & + m_2\beta \sin x_3 + m_2\omega_2 \cos x_3 \\ & & + m_3\beta \sin x_4 + m_3\omega_3 \cos x_4, \\ \dot{x}_2 & = & \omega_1, \\ \dot{x}_3 & = & \omega_2, \\ \dot{x}_4 & = & \omega_3, \end{array}\right. \tag{1}$$

where, $x_0 = \phi \in \mathbb{S}^1$, $x_1 = \dot{\phi} \in \mathbb{R}^1$, $x_2 = \omega_1 t \in \mathbb{S}^1$, $x_3 = \omega_2 t \in \mathbb{S}^1$, and $x_4 = \omega_3 t \in \mathbb{S}^1$. The $\mathbb{S}_1$ denotes 1-dimensional toroidal coordinate system identifying $-\pi$ with π. Parameters β and σ denote normalized angular frequency and detuning, respectively. Parameters m_i and ω_i denote ith input amplitude and angular frequency, respectively. In the following, we fix parameters as follows: $\beta = 0.56, \sigma = 1.3, m_1 = 1, m_2 = 0.03, m_3 = 0.01, \omega_2 = 0.473$, and $\omega_3 = 0.21$. We vary parameter ω_1 and calculate LEs as shown in Fig. 2. Multiplicity of zero LE is three before

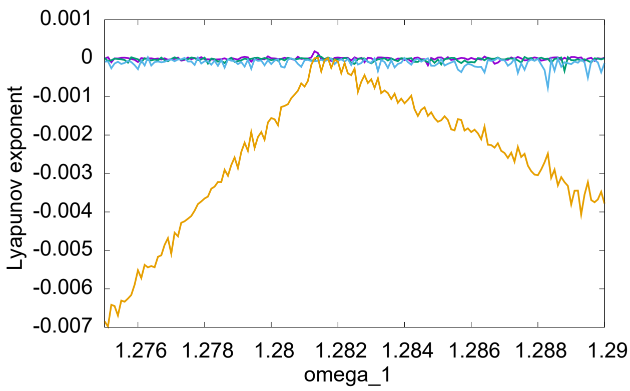
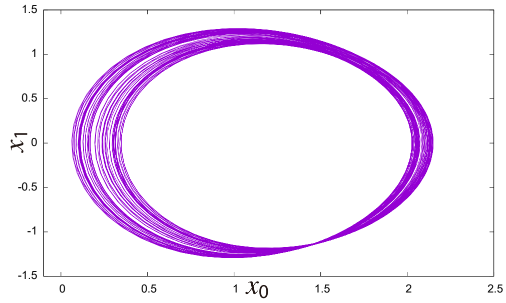
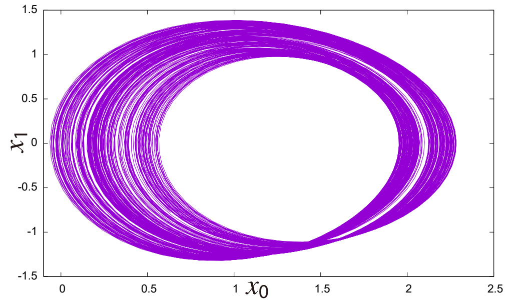


Figure 2: Lyapunov exponents of FT3 in terms of ω_1.

and after the bifurcation point.. Therefore, the attractor is FT3 throughout this figure. The DLE touches zero around $\omega_1 \simeq 1.282$ and some local quasi-periodic bifurcation occurs at this point. Figs. 3(a) and (b) represent FT3 attractor before the bifurcation point at $\omega_1 = 1.286$ and after the bifurcation point at $\omega_1 = 1.275$. To classify this local bi-



(a)



(b)

Figure 3: FT3 attractors at $\omega_1 = 1.286$ before the bifurcation (a) and at $\omega_1 = 1.275$ after the bifurcation point (b).

furcation, we compute DLB before the bifurcation point at $\omega_1 = 1.286$. To judge the type of DLB, we apply Poincare section and codim 2 section to the DLB. As a result, we find that the type of DLB is M type. Therefore, double covering bifurcation of FT3 has occurred at this bifurcation point.

4. Conclusion

We explain how to judge the bifurcation type of 3-torus quasi-periodic solutions. As an example of this, we take PLL circuit equation with three periodic inputs and demonstrate double covering bifurcation of FT3.

Appendix. ST1 and Corresponding DLB

Afterwards, we are succeeded in calculating ST1 and corresponding DLB before the bifurcation associated to Fig. 3 (a). On section $\Sigma_1 = \{x_2 = 0, \dot{x}_2 > 0, x_4 = 0\}$, one 1-turn ST1 in Fig. 4 bifurcates to two 1-turn ST1 in Fig. 5 via component doubling bifurcation of ST1. On the other hand, on section $\Sigma_2 = \{x_3 = 0, \dot{x}_3 > 0, x_4 = 0\}$, one 1-turn ST1 in Fig. 6 bifurcates to one 2-turn ST1 in Fig. 7 via double covering bifurcation of ST1. Fig. 8 shows clearly the A^* type DLB characteristic and Fig. 9 shows the M type characteristic, therefore the FT3 attractor is proved to be the M^3 type DLB.

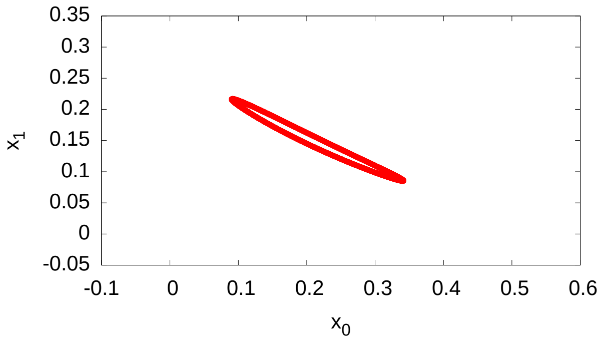


Figure 4: The ST1 of FT3 on section $\Sigma_1 = \{x_2 = 0, \dot{x}_2 > 0, x_4 = 0\}$ at $\omega_1 = 1.286$ before the DC bifurcation.

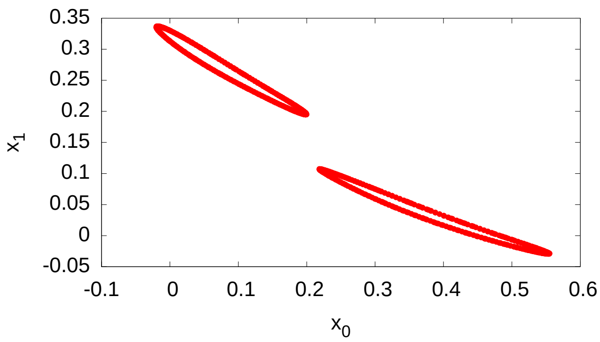


Figure 5: The ST1 of FT3 on section $\Sigma_1 = \{x_2 = 0, \dot{x}_2 > 0, x_4 = 0\}$ at $\omega_1 = 1.275$ after the DC bifurcation.

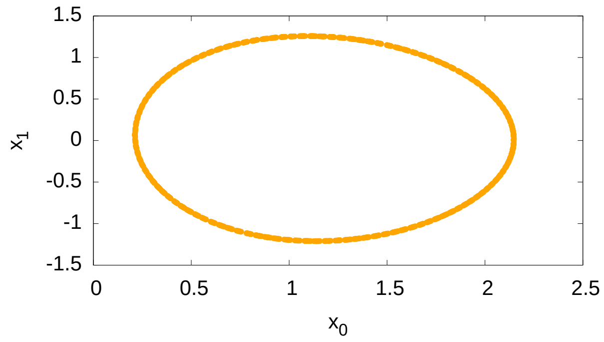


Figure 6: The ST1 of FT3 on section $\Sigma_2 = \{x_3 = 0, \dot{x}_3 > 0, x_4 = 0\}$ at $\omega_1 = 1.286$ before the DC bifurcation.

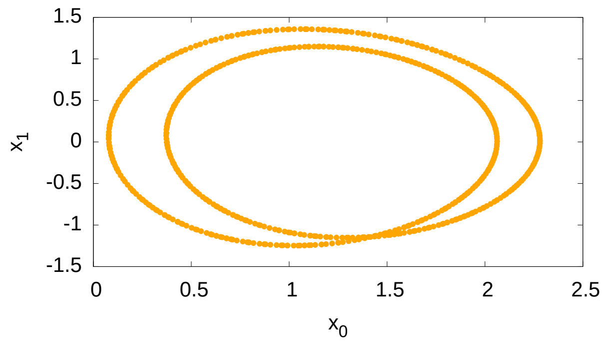


Figure 7: The ST1 of FT3 on section $\Sigma_2 = \{x_3 = 0, \dot{x}_3 > 0, x_4 = 0\}$ at $\omega_1 = 1.275$ after the DC bifurcation.

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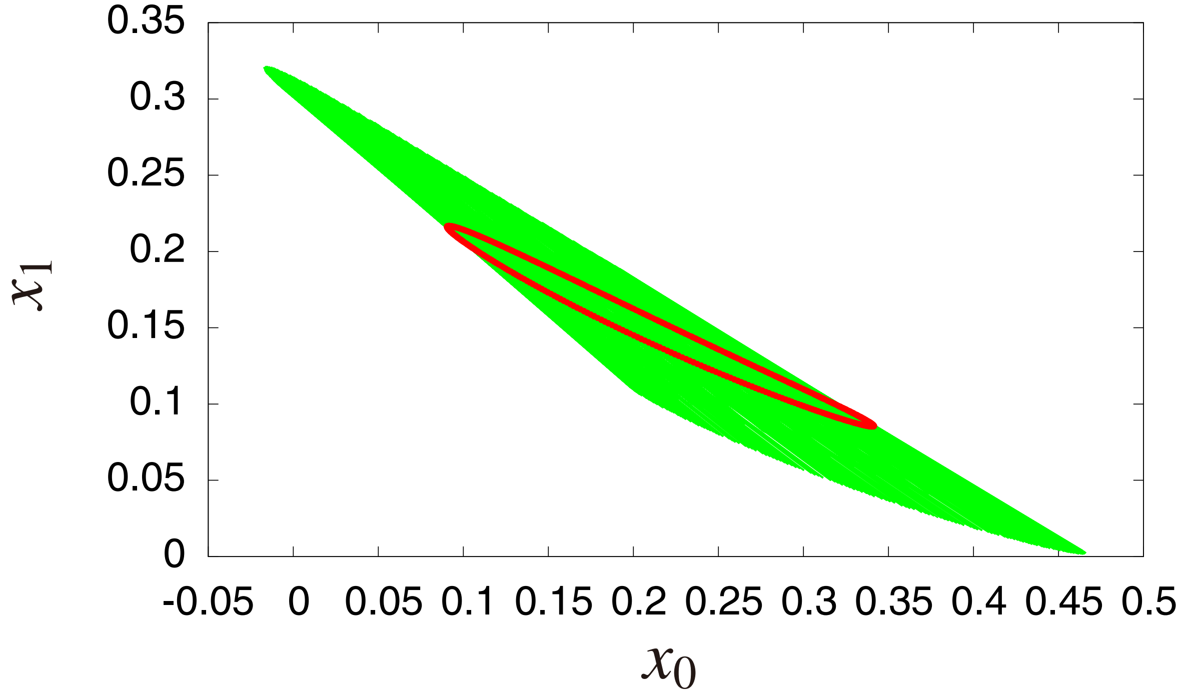

Figure 8: The A^* type DLB of ST1 (green lines) on section $\Sigma_1 = \{x_2 = 0, \dot{x}_2 > 0, x_4 = 0\}$ at $\omega_1 = 1.286$ before the DC bifurcation.

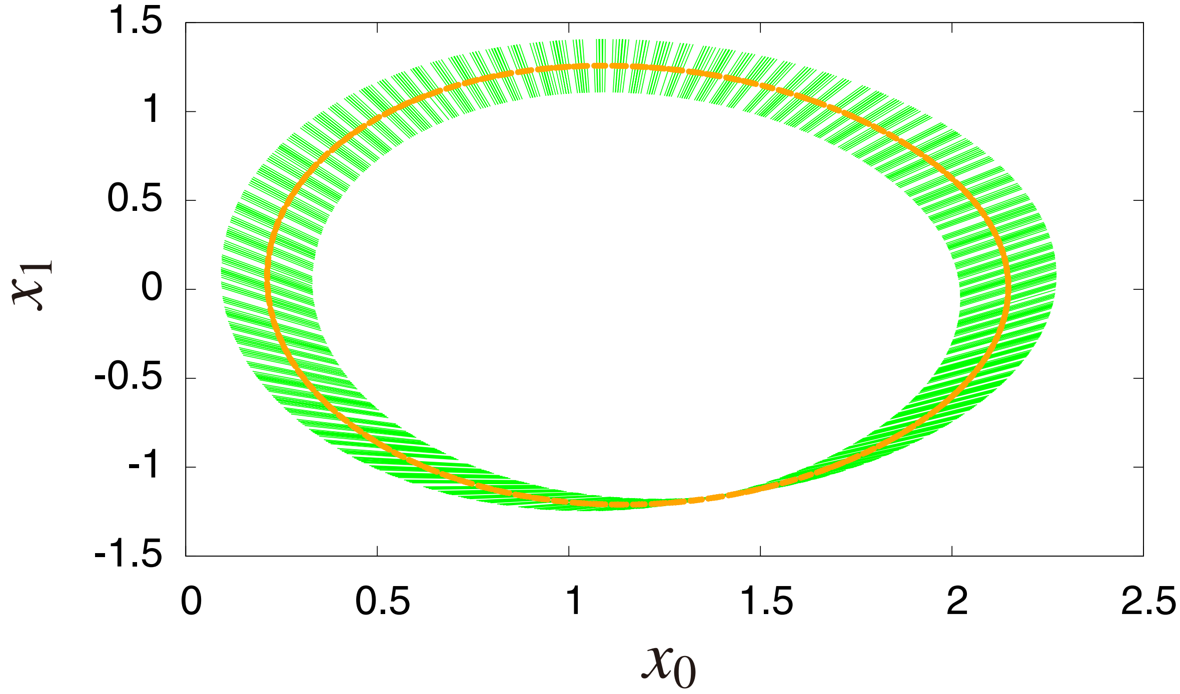

Figure 9: The M type DLB of ST1 (green lines) on section $\Sigma_2 = \{x_3 = 0, \dot{x}_3 > 0, x_4 = 0\}$ at $\omega_1 = 1.286$ before the DC bifurcation.