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MMO-Incrementing Bifurcations in Two Coupled Bonhoeffer-Van der Pol Oscillators

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Abstract—Mixed-mode oscillations (MMOs) have a distinctive waveform in the time series, and consist of large amplitude excursions and small peaks. The MMO-incrementing bifurcations (MMOIBs) trigger an MMO sequence that is followed by another type of an MMO sequence. In this study, we observe MMOIBs in two-coupled Bonhoeffer-van der Pol (BVP) oscillators connected by a resistor by changing a coupling factor. The parameter values of one of the BVP oscillators chosen near a supercritical Andronov-Hopf bifurcation (AHB) point in the absence of connection, whereas those of the other BVP oscillator are chosen near a subcritical AHB point.

1. Introduction

MMOs are nonlinear phenomena, which consist of L large amplitude excursions and s small peaks in the observed time series. To identify various MMOs, we use the notation "L^s". Although the definition of MMOs appears to be ambiguous, the study of MMOs is significant because they are universally observed in slow–fast dynamics. Therefore, MMOs were discovered and analyzed within various systems, and have been a subject of intensive research [1–6]. Moreover, MMO-incrementing bifurcation (MMOIIB) triggers an MMO sequence that, upon varying a parameter, is followed by another type of MMO sequence. That is, MMOIBs that consist of [L^s, $\acute{L}^{\acute{s}} \times n$] ($n = 1, 2, 3, \ldots$) occur many times successively. The complex bifurcations were often observed in actual chemical experiments [7, 8]. Although the generation of MMOIBs appears to be universal in a class of MMOs generating dynamics, the governing equations that describe the chemical experiments may be hardly derived. Furthermore, the study of MMOIBs in dynamical systems, especially for their originating mechanisms, has just begun because they consist of complex MMO-sequences.

In our previous studies, we demonstrate that the successive MMOIBs occur as many times as desired in the Bonhoeffer-van der Pol (BVP) oscillator under a weak periodic perturbation [5, 6]. In particular, we set the parameter values near a subcritical Andronov-Hopf bifurcation (AHB) point with no perturbation. When a weak periodic perturbation is injected to BVP oscillator, MMOIBs can occur successively in both numerical and experimental results.

In this study, we investigate a coupling system that consists of two BVP oscillators connected by a resistor. Because the BVP oscillator play a fundamental role for use in a coupling system [9], the observed phenomena are considered to be important. The parameter values of one of BVP oscillators are chosen near a subcritical AHB point in the absence of connection, whereas those of the other BVP oscillator are chosen near a supercritical AHB point. In this study, we report numerical observations of successive MMOIBs in the two coupled BVP oscillators by changing a coupling factor between the two oscillators. We show that MMOIBs [1^3, $1^4 \times n$], $n = 1, 2, \ldots$, occur many times through the observation of the one-parameter bifurcation diagram.

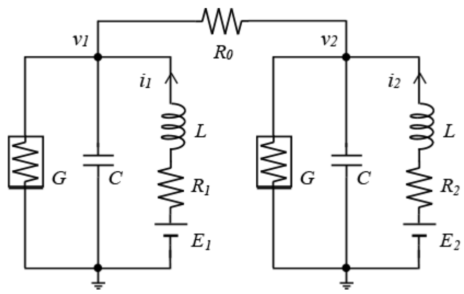
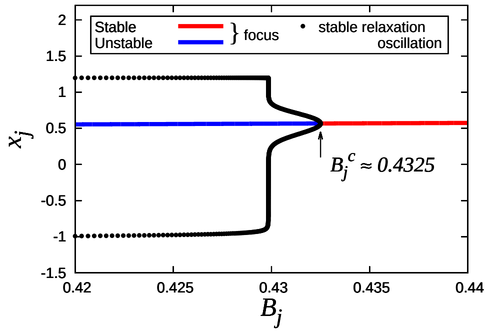


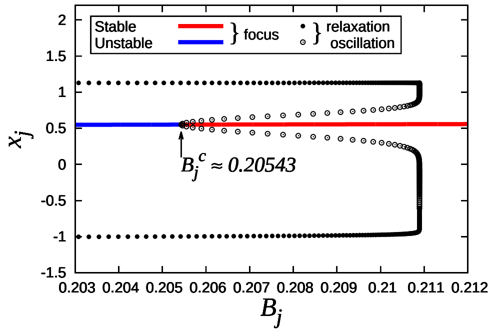
Figure 1: Two BVP oscillators coupled by a resistor.

2. Circuit-setup

Figure 1 shows circuit diagram of two BVP oscillators connected by a resistor R_0. BVP oscillator (O_j, $j = 1, 2$)



(a) $k_j = 0.35$.



(b) $k_j = 0.9$.

Figure 2: Stable and unstable solutions around AHB point in the isolated BVP oscillator for $k_j = 0.35$ and 0.9.

comprises an inductor (L), capacitor (C), resistor (R_j), dc voltage source (E_j), and nonlinear conductance (G). We assume that the voltage–current (v_j–i_j) characteristic of G is a third-order polynomial of the form $i_j = -g_1 v_j + g_3 v_j^3$, where $g_1, g_3 > 0$.

From Kirchhoff's law, the circuit equation of Fig. 1 is written as

$$\begin{aligned} C\frac{dv_1}{dt} &= i_1 - (-g_1 v_1 + g_3 v_1^3) + \frac{1}{R_0}(v_2 - v_1), \\ L\frac{di_1}{dt} &= -v_1 - i_1 R_1 + E_1, \\ C\frac{dv_2}{dt} &= i_2 - (-g_1 v_2 + g_3 v_2^3) + \frac{1}{R_0}(v_1 - v_2), \\ L\frac{di_2}{dt} &= -v_2 - i_2 R_2 + E_2. \end{aligned} \tag{1}$$

Substituting

$$\begin{aligned} &\tau \equiv \frac{t}{Lg_1}, \quad \varepsilon \equiv \frac{C}{g_1^2 L}, \quad k_j \equiv g_1 R_j, \\ &B_j \equiv \sqrt{\frac{g_3}{g_1}} E_j, \quad x_j \equiv \sqrt{\frac{g_3}{g_1}} v_j, \quad y_j \equiv \sqrt{\frac{g_3}{g_1^3}} i_j, \\ &\alpha \equiv (R_0 g_1)^{-1}, \qquad (j = 1, 2), \end{aligned} \tag{2}$$

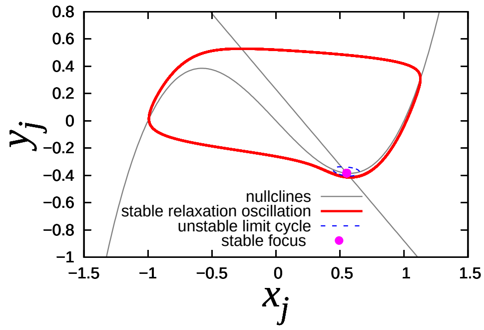


Figure 3: Coexisting stable focus and stable relaxation oscillation of Fig. 2 (b) for $B_j = 0.207$.

into Eq. (1) yields the following normalized equation:

$$\begin{aligned} \varepsilon \dot{x}_1 &= y_1 - (-x_1 + x_1^3) + \alpha(x_2 - x_1), \\ \dot{y}_1 &= -x_1 - k_1 y_1 + B_1, \\ \varepsilon \dot{x}_2 &= y_2 - (-x_2 + x_2^3) + \alpha(x_1 - x_2), \\ \dot{y}_2 &= -x_2 - k_2 y_2 + B_2, \quad \text{where } \frac{d}{d\tau} = \cdot\,, \end{aligned} \tag{3}$$

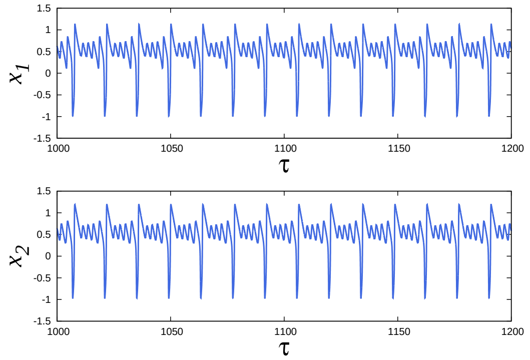
where ε is a small parameter that corresponds to the small capacitance C. We assume $\varepsilon = 0.1$ throughout this study. The parameters α, k_j and B_j correspond to the coupling factor R_0, the resistance R_j and the dc bias voltage E_j, respectively ($j = 1, 2$).

A stable focus exists when B_j ($j = 1, 2$) is chosen as large value, and when the two BVP oscillators are not connected with each other, *i.e.*, $\alpha = 0$. This stable focus becomes unstable via an AHB point (the bifurcation parameter value is indicated by B_j^c). Furthermore, the AHB is a supercritical for small k_j, whereas it becomes subcritical for larger k_j. Figures 2 (a) and (b) show the structures around the super and subcritical AHB points for $k_j = 0.35$ and 0.9, respectively. In particular, for $k_j = 0.9$, the AHB is subcritical, and a stable focus and the stable relaxation oscillation coexist in close proximity in the phase space near $B_j = B_j^c$ as shown in Fig. 3.

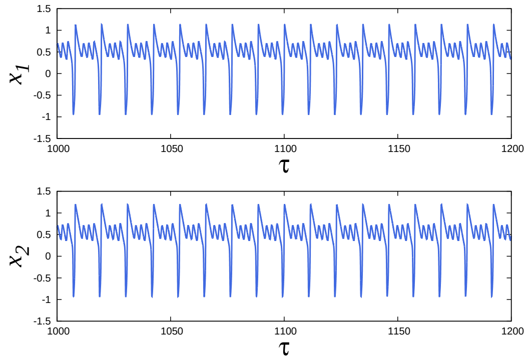
In the following results, we use $k_1 = 0.9, B_1 = 0.207, k_2 = 0.35$, and $B_2 = 0.428$, and employ the coupling factor α as a control parameter. In order to calculate Eq. (3) with the initial condition $x_j = y_j = 0, j = 1, 2$, we use a fourth-order Runge-Kutta method with the step size 0.01.

3. MMO-incrementing bifurcations

Figure 4 (a) shows MMOs 1^4, in which x_j, $j = 1, 2$ undergoes one large excursion followed by four successive small peaks in the observed time series in the two coupled BVP oscillators for $\alpha = 0.202$. Moreover, when α is decreased, different MMO sequence 1^3 is observed as



(a) 1^4 (α = 0.202).



(b) 1^3 (α = 0.182).

Figure 4: MMOs in the two coupled BVP oscillators.

depicted in Fig. 4 (b). Thus, MMOs 1^s, $s = 1, 2, \ldots$ are naturally observed in the two coupled BVP oscillators.

We are now interested in the phenomena between two neighboring MMOs. When α is slightly decreased from 0.202, more complex MMO sequence is observed. Figure 5 (a) shows the complex MMO sequence in which 1^3 and 1^4 appear alternately. For smaller α, MMOIBs, which triggers an MMO sequence that upon varying a parameter is followed by another type of MMO sequence, occur [5]. Figure 5 (b) depicts MMOIBs in which MMO sequence 1^3 is added to that of Fig. 5(a). In the same manner, $[1^4, 1^3 \times 3]$ appears for $\alpha = 0.188$ as shown in Fig. 5 (c). Although, at first glance, the two BVP oscillators are synchronized with in-phase, the phase relationship between the two oscillators is complex as shown in Fig. 5 (d). To investigate MMOIBs more in detail, we calculate a one-parameter bifurcation diagram. Figure 6 (a) shows the one-parameter bifurcation diagram for α decreasing from 0.202, where Poincaré mapped points of the values of x_2 are plotted. We define Poincaré section as $x_1 = 1/\sqrt{3}$, and map points when the flow penetrates the hyperplane from the negative side to the positive side. Furthermore, Fig. 6 (b) is the magnified figure of Fig. 6 (a) for $0.184 \leqq \alpha \leqq 0.185$. From the figure, it is seen that MMOIBs occur many times as α decreases. One of the characteristic features of MMOs is a period-adding. In the same manner, MMOIBs occur successively as shown in Fig. 6.

4. Concluding remarks

We investigated the two coupled BVP oscillators, in which the parameter values of one of the BVP oscillators were chosen near a supercritical AHB point for $\alpha = 0$, whereas those of the other BVP oscillator were chosen near a subcritical AHB point. It is remarkable that MMOIBs occur many times. The detailed initiating mechanism of MMOIBs in the two BVP oscillators remains a subject for a future study.

References

[1] M. Brons, T. J. Kaper, and H. G. Rotstein, "Introduction to focus issue: Mixed mode oscillations : experiment, computation, and analysis," Chaos, vol. 18, 015101, 2008.

[2] M. Krupa, N. Popovic, and N. Kopel, "Mixed-mode oscillations in three time-scale systems: A prototypical example," SIAM J. Applied Dyn. Sys., vol. 7, pp. 361–420, 2008.

[3] J.G. Freire, J.A.C. Gallas, "Stern-Brocot trees in cascades of mixed-mode oscillations and canards in the extended Bonhoeffer.van der Pol and the FitzHugh-Nagumo models of excitable systems," Phys. Lett. A, vol. 375, pp. 1097–1103, 2011.

[4] E. Kutafina, "Mixed mode oscillations in the Bonhoeffer-van der Pol oscillator with weak periodic perturbation," Comp. Appl. Math., 2013 (web online).

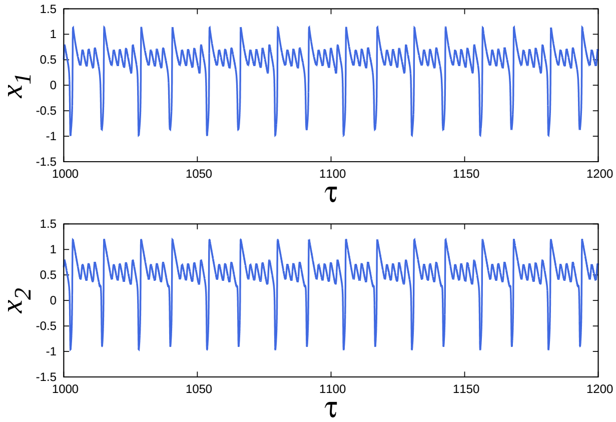
[5] K. Shimizu, Y. Saito, M. Sekikawa, and N. Inaba, "Complex mixed-mode oscillations in a Bonhoeffer-van der Pol oscillator under weak periodic perturbation," Physica D, vol. 241, pp. 1518–1526, 2012.

[6] K. Shimizu, M. Sekikawa, and N. Inaba, "Experimental study of complex mixed-mode oscillations generated in a Bonhoeffer-van der Pol oscillator under weak periodic perturbation," Chaos, vol. 25, 023105, 2015.

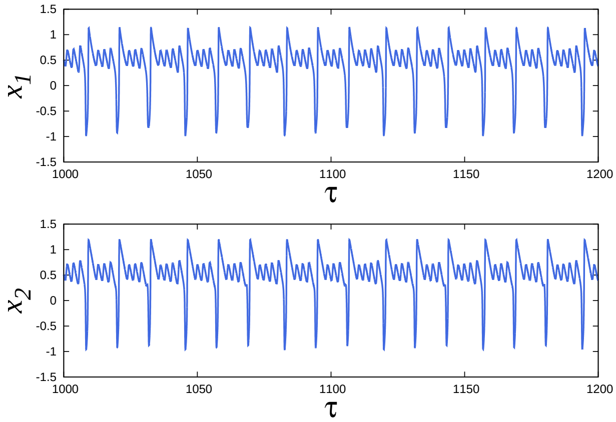
[7] J. Maselko and H. L. Swinney, "Complex periodic oscillations and Farey arithmetic in the BelousovZhabotinskii reaction," J. Chem. Phys., vol. 85, pp. 6430–6441, 1986.

[8] F. N. Albahadily, J. Ringland, and M. Schell, "Mixed-mode oscillations in an electrochemical system. I. A Farey sequence which does not occur on a torus," J. Chem. Phys., vol. 90, pp. 813–821, 1989.

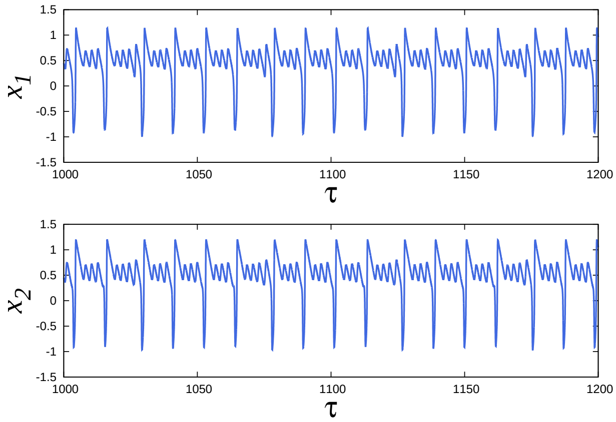
[9] T. Ueta, H. Miyazaki, T. Kousaka, and H. Kawakami, "Bifurcation and chaos in coupled BVP oscillators," Int. J. Bifurcation Chaos, vol.14, no.4, pp.1305-1324, 2004.



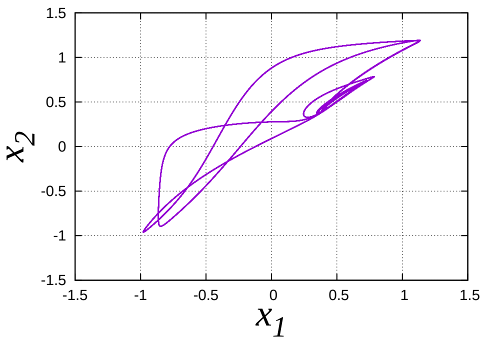
(a) $[1^4, 1^3 \times 1]$ ($\alpha = 0.196$).



(b) $[1^4, 1^3 \times 2]$ ($\alpha = 0.191$).

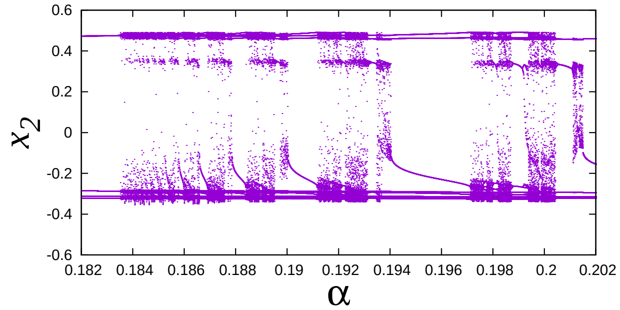


(c) $[1^4, 1^3 \times 3]$ ($\alpha = 0.188$).

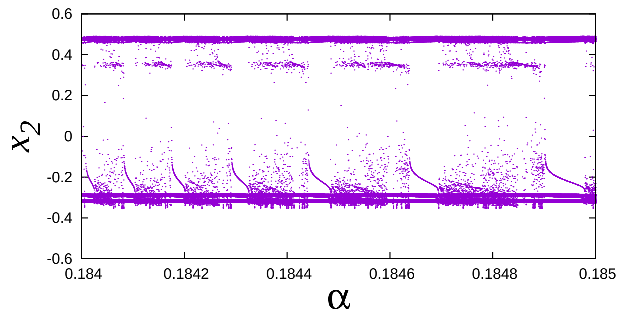


(d) Phase relationship between the two oscillators of (a).

Figure 5: Complex MMO sequence.



(a)



(b) Magnified figure of (a).

Figure 6: One-parameter bifurcation diagram for $0.182 \leqq \alpha \leqq 0.202$.