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Improvement of the Multiple Shooting Method for Stability Analysis of Periodic Orbits of Ordinary Differential Equations

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Abstract—This paper describes a numerical method to examine stability of periodic orbits of ordinary differential equations. The multiple shooting is one of the commonly used methods for numerical stability analysis, but this method may produce incorrect results under some conditions. We focus on a certain property of solutions of equations, and propose an improved method of the multiple shooting. A numerical example shows effectiveness of the proposed method.

1. Introduction

This paper studies stability of periodic orbits of ordinary differential equations given by

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{f}(t, \boldsymbol{x}), \quad \boldsymbol{f}(t, \boldsymbol{x}) = \boldsymbol{f}(t + T, \boldsymbol{x}), \tag{1}$$

where $\boldsymbol{x} \in \mathbb{R}^N$ and $\boldsymbol{f} : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is of class C^2. A periodic solution of (1) with the period T and $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$ can be expressed as $\boldsymbol{x}(t) = \varphi^t(\boldsymbol{x}_0)$, and satisfies $\varphi^T(\boldsymbol{x}_0) = \boldsymbol{x}_0$. Stability of this periodic orbit is determined using the corresponding variational equations given by

$$\frac{\mathrm{d}\boldsymbol{X}}{\mathrm{d}t} = \left.\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\right|_{\boldsymbol{x}=\varphi^t(\boldsymbol{x}_0)} \boldsymbol{X} \quad \text{with } \boldsymbol{X}(t = t_0) = I, \tag{2}$$

where $\boldsymbol{X} \in \mathbb{R}^{N\times N}$ and I denotes the identity matrix. Eigenvalues of the matrix solution $X(t_0 + T)$ determine the stability.

There are some numerical methods to simultaneously solve equations (1) and (2) such as the finite difference method and the collocation method [1]. In this work, we use the multiple shooting method because it is relatively straightforward and commonly used. Also the idea of this method has been applied to numerical verification of periodic orbits of ordinary differential equations [2] [3]. On the other hand, it has been reported that the multiple shooting method can produce incorrect results under some conditions[4] [5] [6].

In this work, we try to improve the multiple shooting method for (1) and (2). Section 2 summarizes the multiple shooting method and its drawbacks which can cause some troubles. Section 3 focuses on a certain property of solutions of (1) and (2), and propose an improved method of the multiple shooting. Section 4 compares these two methods using a numerical example.

2. The multiple shooting method for periodic orbits [1]

The basic idea of the multiple shooting method for a periodic orbit $\boldsymbol{x}(t) = \varphi^t(\boldsymbol{x}_0)$ of (1) with the period T is to divide the orbit into $M + 1$ parts such as

$$\boldsymbol{x}_{k+1} = \boldsymbol{x}(t_{k+1}) = \varphi^{t_{k+1}-t_k}(\boldsymbol{x}_k) \qquad (k = 0, 1, \cdots, M), \tag{3}$$

where $t_0 < t_1 < \cdots < t_M < t_{M+1} = t_0 + T$ and $\boldsymbol{x}_{M+1} = \boldsymbol{x}_0$, and to iteratively obtain approximate solutions of $\{t_k, \boldsymbol{x}_k\}_{k=0,1,\cdots,M}$ using Newton's method such that (3) is satisfied. In (3), $\varphi^{t_{k+1}-t_k}(\boldsymbol{x}_k)$ can be numerically solved using the Runge-Kutta method.

Writing $t = t_k + \tau$, we can express the corresponding variational equations as

$$\frac{\mathrm{d}X_k}{\mathrm{d}\tau} = \left.\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\right|_{t=t_k+\tau,\, \boldsymbol{x}=\varphi^\tau(\boldsymbol{x}_k)} X_k(\tau) \qquad \text{with } X_k(0) = I, \tag{4}$$

for $k = 0, 1, \cdots, M$ where $X_k(\tau) \in \mathbb{R}^{N\times N}$. Stability of the periodic orbit is determined by eigenvalues of the matrix $X(t_0 + T)$ given by

$$X(t_0 + T) = X_M(h_M)\, X_{M-1}(h_{M-1}) \cdots X_0(h_0), \tag{5}$$

where $h_k = t_{k+1} - t_k$. The eigenvalues are called Floquet multipliers. We can obtain approximate solutions of $X_k(h_k)$ using the Runge-Kutta method.

Hereafter we call this standard multiple shooting method "Method 1". As mentioned in the introduction, this "Method 1" produces reliable results in most cases and the computational cost is relatively low, but may give incorrect results under some conditions. We consider that the numerical method for the variational equations (4) may cause some troubles. It is because (4) is solved as an initial value problem and the errors are not controlled. In the next section, we try to overcome this problem using a certain property of solutions of (4).

3. Improvement of the multiple shooting method

The map φ^t of a solution $\boldsymbol{x}(t) = \varphi^t(\boldsymbol{x}_0)$ of ordinary differential equations has the following property

$$\varphi^{t+s} \;=\; \varphi^t \circ \varphi^s \qquad (t, s \in \mathbb{R})\,. \tag{6}$$

From this property, we can get

$$\varphi^{h_k - s_k}(\boldsymbol{x}_k) \;=\; \varphi^{-s_k}(\varphi^{h_k}(\boldsymbol{x}_k)) \;=\; \varphi^{-s_k}(\boldsymbol{x}_{k+1})\,, \tag{7}$$

for $k = 0, 1, \cdots, M$. In this work, we focus on this property, namely

$$\boldsymbol{u}_k(\boldsymbol{x}_k, \boldsymbol{x}_{k+1}, h_k, s_k) \;:=\; \varphi^{h_k - s_k}(\boldsymbol{x}_k) - \varphi^{-s_k}(\boldsymbol{x}_{k+1}) \;=\; \boldsymbol{0}\,. \tag{8}$$

Also a solution of the variational equation (4) satisfies the above property. Writing the solution as $X_k(\tau) = \psi_k^\tau(X_k(0))$ where the map ψ_k^τ satisfies $\psi_k^{\tau+\sigma} = \psi_k^\tau \circ \psi_k^\sigma$ $(\tau, \sigma \in \mathbb{R})$, we can express the property as

$$\begin{aligned} &V_k(X_k(0), X_k(h_k), h_k, s_k) \\ &\quad:=\; \psi^{h_k - s_k}(X_k(0)) - \psi^{-s_k}(X_k(h_k)) \;=\; O\,. \end{aligned} \tag{9}$$

where O denotes the null matrix.

Approximate solutions $\tilde{\boldsymbol{u}}_k$ and $\tilde{V}_k$ of $\boldsymbol{u}_k$ and V_k in equations (8) and (9) can be numerically obtained using the Runge-Kutta method and, in general, $\tilde{\boldsymbol{u}}_k \neq \boldsymbol{0}$ and $\tilde{V}_k \neq O$. The idea of the proposed method is to modify the approximate solutions $\tilde{t}_k, \tilde{\boldsymbol{x}}_k, \tilde{X}_k(\tilde{h}_k)$ using Newton's method such that (8) and (9) are simultaneously satisfied with enough accuracy. We call this method "Method 2". Then the convergence conditions of Newton's method are given by

$$\max_{0\le k\le M} |\tilde{\boldsymbol{u}}_k| \;<\; \delta_1 \;\text{ and }\; \max_{0\le k\le M} \|\tilde{V}_k\| \;<\; \delta_2\,, \tag{10}$$

with

$$\|\tilde{V}_k\| = \|(\tilde{\boldsymbol{v}}_1 \tilde{\boldsymbol{v}}_2 \cdots \tilde{\boldsymbol{v}}_N)\| \;:=\; \max_{1\le j\le N} |\tilde{\boldsymbol{v}}_j|\,. \tag{11}$$

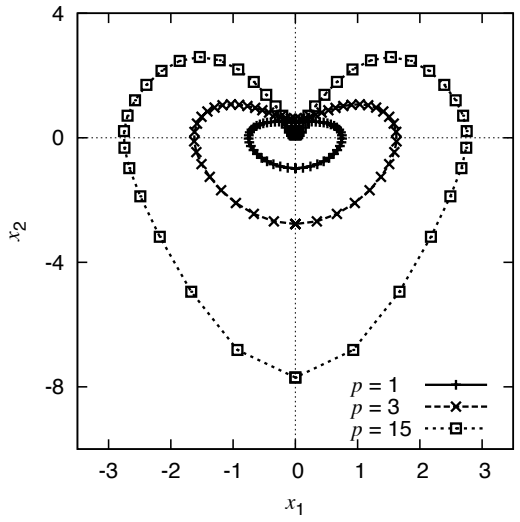
where $\tilde{\boldsymbol{v}}_j$ denotes the j-th column vector of the matrix $\tilde{V}_k$. These conditions can be used as one of indices for accuracy of solutions.

4. Numerical example

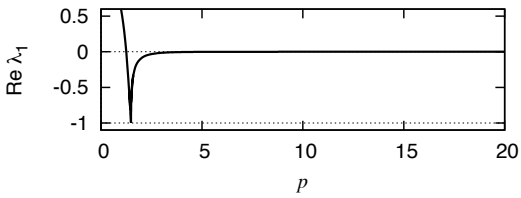
This section compares computed results using the standard multiple shooting method "Method 1" in section 2 and using the proposed method "Method 2" in section 3 for a nonlinear Mathieu equation [7] given by

$$\begin{aligned} \frac{\mathrm{d}x_1}{\mathrm{d}t} &= x_2\,, \\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= -(1 + p\cos t)\,\sin x_1\,, \end{aligned} \tag{12}$$

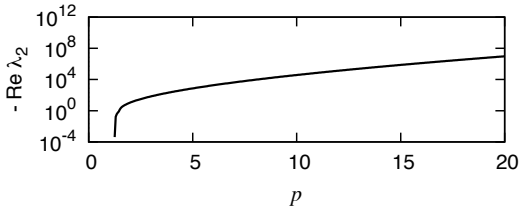
where $p \ge 0$ is a parameter. This is the equation of motion of a periodically perturbed pendulum. Note that the origin is an equilibrium. Figure 1 shows computed results of periodic orbits with the period $T = 2\pi$ for $1 \le p \le 15$, using



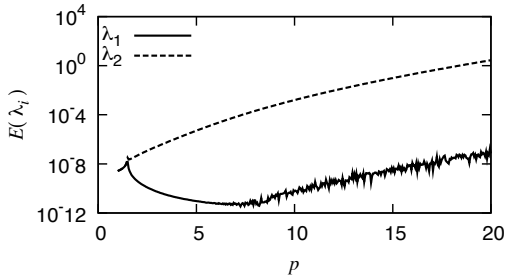
(a) Periodic orbits in the state space for p = 1,3 and 15.



(b) The real part of λ_1



(c) The real part of λ_2



(d) Errors of λ_1 and λ_2

Figure 1: Computed results of the periodic orbits of the Mathieu equation (12) and the Floquet multipliers λ_1 and λ_2 by "Method 2" in Section 3. $E(\lambda_i) = |\lambda_i - \hat{\lambda}_i|$ $(i = 1, 2)$ and $\hat{\lambda}_i$: the best approximate solutions of λ_i computed using "Method 1" with $\Delta t = 0.0025$ for the 4th order Runge-Kutta method and $\delta_1 = 10^{-14}$ for Newton's method.

the proposed method "Method 2" with $M+1=50$. The convergence conditions (10) of Newton's method were set to $\delta_1 = \delta_2 = 1 \times 10^{-6}$, and the time increment Δt of the 4th order Runge-Kutta method $\Delta t = 0.01$, respectively. Since (12) is non-autonomous and the period $T = 2\pi$ is given, $t_k = \frac{T}{M}k$ and $h_k = t_{k+1} - t_k = \frac{T}{M}$ $(k = 0, 1, \cdots, M)$ are fixed. For simplicity, s_k was set to $s_k = \frac{h_k}{2} = \frac{T}{2M}$.

Figure 1 (a) shows that the orbit gets closer to the equilibrium with increase of p. Figures 1 (b) and (c) show the real parts of the computed Floquet multipliers λ_1 and λ_2, respectively. It is found that $\mathrm{Re}\lambda_1$ approaches zero and $|\lambda_2|$ becomes considerably large with increase of p. It should be noted that the imaginary parts of λ_1 and λ_2 are zero for p is larger than about 1.49. Thus the matrix $X(t_0+T)$ of (5) becomes almost singular with increase of p.

Shapes of the periodic orbits computed by "Method 2" were almost the same as those by "Method 1". On the other hand, we could find some differences of computed Floquet multipliers between these two methods. In order to examine these differences, we used the computed results using "Method 1" with the severe convergence condition $\delta_1 = 1 \times 10^{-14}$ of Newton's method and the fine time increment $\Delta t = 0.0025$ of the Runge-Kutta method, as the best approximate solution. In this paper, we call "Error" by the difference between the computed and the best approximate solution. Figure 1(d) shows "Error"s $E(\lambda_i) = |\lambda_i - \hat{\lambda}_i|$ $(i = 1, 2)$ of the Floquet multipliers λ_1 and λ_2 computed by "Method 2". Here $\hat{\lambda}_i$ denotes the best approximate results. We can see that the errors $E(\lambda_1)$ and $E(\lambda_2)$ grow with increase of p.

Figures 2 (a)~(d) compare the errors of (8) and (9), namely $\max_{0\le k\le M} |\tilde{\boldsymbol{u}}_k|$ and $\max_{0\le k\le M} \|\tilde{V}_k\|$, during iterations of Newton's method in "Method 1" and "Method 2". Figures 2 (a) and (b) show that both methods work very well for reduction of the errors of (8) about solutions of the ordinary differential equations. On the other hand, Figures 2 (c) and (d) indicate that "Method 2" can control and significantly reduce the errors of (9) about solutions of the variational equations, but that "Method 1" does not do that. These results suggest that "Method 2" improves "Method 1".

Figures 2 (e)~(h) compare the "Error"s of the Floquet multipliers $E(\lambda_1)$ and $E(\lambda_2)$ computed by the two methods. We can find slight improvement of "Method 2" for $\Delta t = 0.1$, but no remarkable differences for $\Delta t = 0.01$.

In this work, we have tried to improve the multiple shooting method using the condition (6). We can derive some critical properties of dynamical systems from this condition $\varphi^{t+s} = \varphi^t \circ \varphi^s$ $(t, s \in \mathbb{R})$. That is the reason why we focus on (6). Although we could not find crucial effectiveness in this simple example, in order to investigate significance of this condition (6) in numerical stability analysis, we have to apply the proposed method to some other systems for which numerical difficulties have been reported.

5. Conclusion

This work has considered the numerical method for stability of periodic orbits of ordinary differential equations. In particular, we have tried to improve the multiple shooting method using a certain property (6) of solutions of systems. The basic idea is to iteratively modify approximate solutions of ordinary differential equations and the corresponding variational equations using (8) and (9) derived from (6).

The proposed method improves the original multiple shooting method in the sense that the two conditions (8) and (9) are simultaneously satisfied with enough accuracy, as shown in the numerical example. In order to study the role of (6) in the numerical stability analysis, we plan to perform extensive numerical experiments in future.

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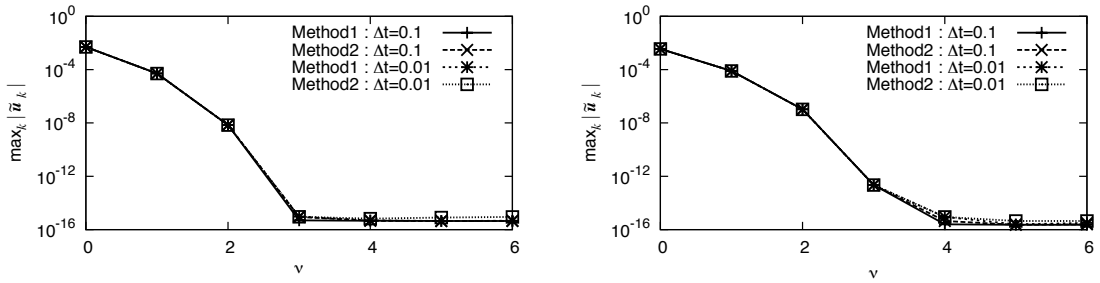
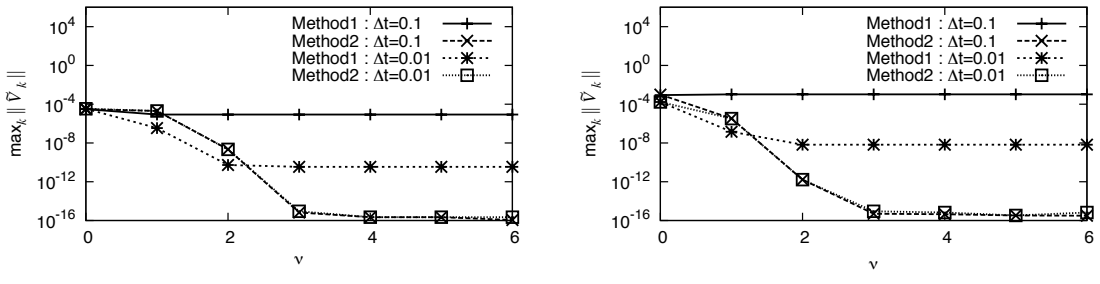
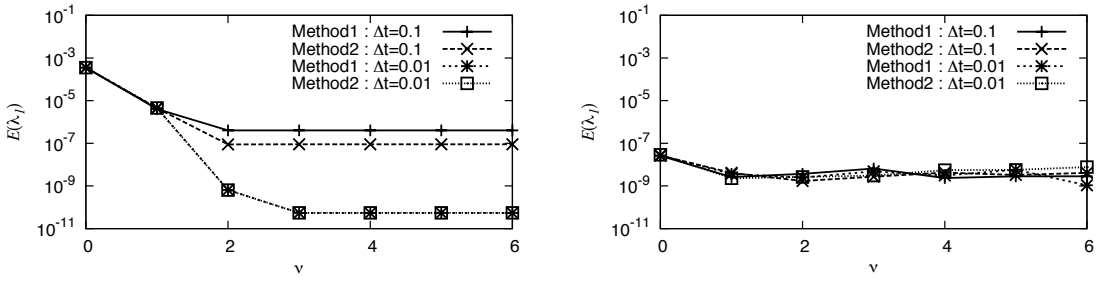


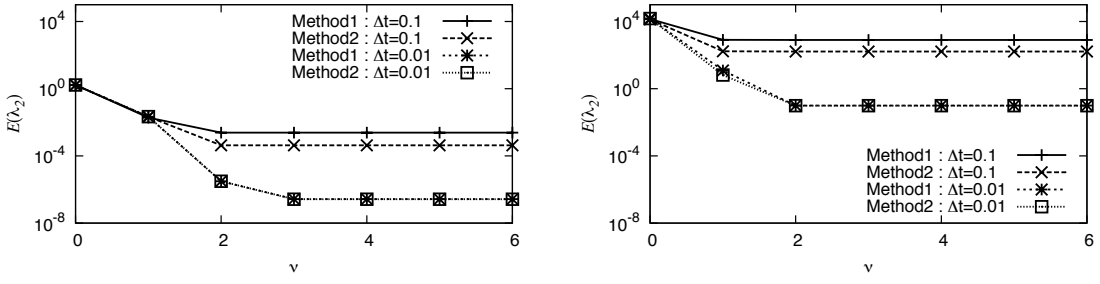
(a) $\max_{0\le k\le M}|\tilde{\boldsymbol{u}}_k|$, $p = 3$ (b) $\max_{0\le k\le M}|\tilde{\boldsymbol{u}}_k|$, $p = 15$



(c) $\max_{0\le k\le M}\|\tilde{V}_k\|$, $p = 3$ (d) $\max_{0\le k\le M}\|\tilde{V}_k\|$, $p = 15$



(e) $E(\lambda_1)$, $p = 3$ (f) $E(\lambda_1)$, $p = 15$



(g) $E(\lambda_2)$, $p = 3$ (h) $E(\lambda_2)$, $p = 15$

Figure 2: Comparison of the errors of "Method 1" and "Method 2". (a)~(d) compare the errors of the conditions (8) and (9), and (e)~(h) the errors of the Floquet multipliers for the Mathieu equation (12). ν : the iteration number of Newton's method, Δt : the time increment of the Runge-Kutta, $\boldsymbol{u}_k$: see (8), V_k : see (9), $E(\lambda_i)$, $(i = 1, 2)$: see caption of Figure 1, and $\hat{\lambda}_i$: the best approximate solutions $(\hat{\lambda}_1, \hat{\lambda}_2) = (-1.4439 \times 10^{-2}, -69.254)$, for $p = 3$ and $(\hat{\lambda}_1, \hat{\lambda}_2) = (-1.3404 \times 10^{-6}, -7.4617 \times 10^{5})$ for $p = 15$.