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Analysis of aperiodic oscillations induced by periodic oscillators

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Abstract—With changing the strengths of connections as a parameter, we examined the oscillatory states of the unidirectionally connected Wilson-Cowan oscillators models. We observed aperiodic oscillations as well as periodic oscillations, which depend on the parameter values. Numerical analysis suggests that the aperiodic oscillation is due to the interactions between the bifurcation phenomena of itself and oscillatory external inputs.

1. Introduction

Wilson-Cowan oscillator model is a well-known neural oscillator model, which expresses excitatory and inhibitory firing rates of a population of neurons [1]. The interactions of excitatory and inhibitory populations can be sometimes regarded as local behavior of the brain. Thus Wilson-Cowan oscillator is widely used to compose neural network models (e.g. associative memory, visual cortex, etc. [2, 3, 4, 5]).

The properties of bifurcation of single Wilson-Cowan oscillator have been reported [6], and their weakly connected models are also well investigated [7]. Among them, it has been reported that chaotic oscillation occurs in reciprocally connected Wilson-Cowan oscillators [8], and suggested that single Wilson-Cowan oscillator may also give rise to aperiodic oscillation [7]. However, little attention has been paid to the cause of aperiodic or chaotic oscillations so far.

The purpose of our study is to reveal the mechanism of occurrence and variation of aperiodic oscillations in the coupled Wilson-Cowan oscillator. We connected two oscillators unidirectionally for simplicity, yet such unidirectional connections are often used, for instance, in a chain model [7, 9, 10]. And we examined the mechanism of oscillation with changing strengths of connections.

2. Methods

2.1. Wilson-Cowan oscillator model

Wilson-Cowan oscillator is described by the following differential equations:

$$\dot{E} = -E + (1 - E)\,\mathcal{S}_e\,(c_1 E - c_2 I + P)\,,$$
$$\dot{I} = -I + (1 - I)\,\mathcal{S}_i\,(c_3 E - c_4 I + Q)\,,$$

where E and I denote mean firing rates of excitatory and inhibitory neurons respectively, c_1, c_2, c_3, c_4 are strengths of connections, and P, Q are external inputs (Fig.1). $\mathcal{S}_e(x)$ and $\mathcal{S}_i(x)$ are sigmoid functions as follows:

$$\mathcal{S}_e(x) = \frac{1}{1+\exp(-a_e(x-\theta_e))} - \frac{1}{1+\exp(a_e\theta_e)},$$

$$\mathcal{S}_i(x) = \frac{1}{1+\exp(-a_i(x-\theta_i))} - \frac{1}{1+\exp(a_i\theta_i)}.$$

In this study, $c_1, c_2, c_3, c_4, a_e, \theta_e, a_i$, and θ_i are constant parameters. We set the values as follows: $c_1 = 16.0$, $c_2 = 12.0$, $c_3 = 15.0$, $c_4 = 3.0$, $a_e = 1.3$, $\theta_e = 4.0$, $a_i = 2.0$, and $\theta_i = 3.7$ respectively, following the parameter values in Wilson & Cowan [1].

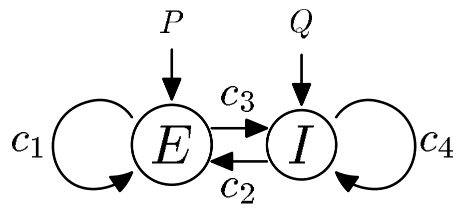


Figure 1: Wilson-Cowan oscillator model.

The bifurcation phenomena of a single Wilson-Cowan oscillator is known to occur depending on P and Q values [7] . Sets of bifurcation points in (P, Q) space are shown in Fig.2, in which the red line denotes the Andronov-Hopf bifurcation points and the green line denotes the saddle-node bifurcation points. For convenience, we call the region 2 “oscillatory region” because of its oscillatory property, and we call region 1 and 3 “convergent region”.

2.2. Connected oscillators model

In this study, we analyzed two Wilson-Cowan oscillators which are connected unidirectionally (Fig.3). One of the oscillators sends signals to the other. Thus one is a sender, and the other is a receiver. Let the sender oscillate by setting the external input in the oscillatory region. We assume that the receiver does not oscillate at initial state. The strengths of connections between oscillators are α and β. We refer the sender to “oscillator 1”, and the receiver to “oscillator 2 ”, for convenience.

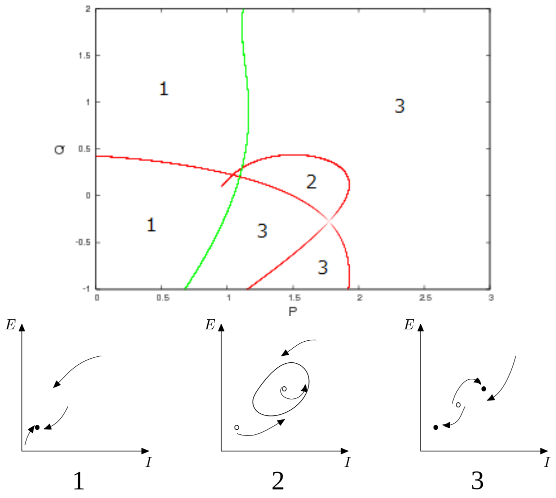


Figure 2: Bifurcation points of Wilson-Cowan oscillator in (P, Q) space (top) and schematic three types of phase space (bottom). Each number of the bottom figures corresponds to those in the top figure. The solid black circle and open circle denote stable and unstable equilibria respectively. The closed trajectory is a stable limit cycle.

Therefore oscillator 1 and 2 are denoted by the following equations:

$$\begin{cases} \dot{E}_1 &= -E_1 + (1 - E_1)\, \mathcal{S}_e\,(c_1 E_1 - c_2 I_1 + P_1)\,, \\ \dot{I}_1 &= -I_1 + (1 - I_1)\, \mathcal{S}_i\,(c_3 E_1 - c_4 I_1 + Q_1)\,, \end{cases} \tag{1}$$

$$\begin{cases} \dot{E}_2 &= -E_2 + (1 - E_2)\, \mathcal{S}_e\,(c_1 E_2 - c_2 I_2 + \alpha E_1)\,, \\ \dot{I}_2 &= -I_2 + (1 - I_2)\, \mathcal{S}_i\,(c_3 E_2 - c_4 I_2 + \beta E_1)\,. \end{cases} \tag{2}$$

Thus we assume that the inputs to the oscillator 2 from the oscillator 1 are represented by $(P, Q) = (\alpha E_1,\ \beta E_1)$. We investigate the change of oscillatory mode of this oscillator with changing parameter α and β.

The external inputs P_1 and Q_1 of the oscillator 1 are set to $P_1 = 1.9$ and $Q_1 = 0.0$ in the oscillatory region respectively so that it oscillates.

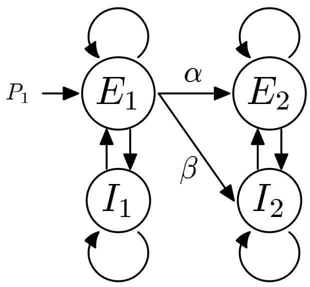


Figure 3: Unidirectionally connected Wilson-Cowan neurons investigated in this study.

3. Results and Discussion

3.1. Periodic oscillation

The oscillator 2 oscillates periodically in a broad range of parameter space (Fig.4). Average amplitude of E_2 increases as parameter α increases, and decreases as parameter β increases in periodic states.

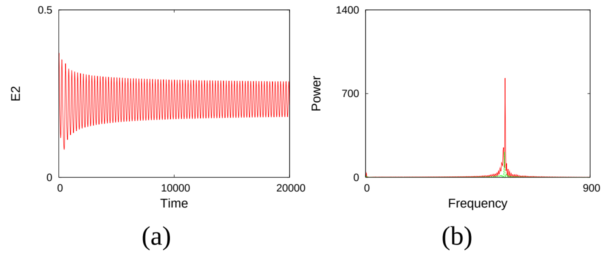


Figure 4: (a) Periodic oscillation of E_2 when $\alpha = 8.00, \beta = 2.00$. (b) The power spectrum of E_2. Green line is the spectrum of E_1, and red line is the spectrum of E_2. They are almost correspondent.

3.2. Aperiodic oscillation

The oscillator 2 shows aperiodic oscillations at particular parameter values (Fig.5). The parameter region corresponding to aperiodic states is, however, much smaller than that in a periodic state.

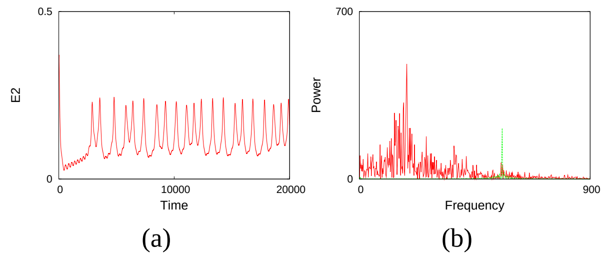


Figure 5: (a) Aperiodic oscillation of E_2 when $\alpha = 4.47, \beta = 0.90$. (b) The power spectrum of E_2. Green line is the spectrum of E_1, and red line is the spectrum of E_2.

3.3. Background of periodic and aperiodic oscillation

We discuss causes of periodic and aperiodic oscillation.

Let us consider the cause of periodic oscillation of the oscillator 2. Wilson-Cowan oscillator has a parameter region in which firing rates converge on a constant value. This convergent region corresponds to the rest area of oscillatory region (region 2 in Fig.2). The oscillator 2 travels over different convergent states since the external inputs oscillate in $(P, Q) = (\alpha E_1,\ \beta E_1)$ space (Fig.2). Oscillation of αE_1 and βE_1 forces the oscillator 2 to transverse various convergent points as time goes on. The emergent periodic oscillation will be due to smooth oscillation of the convergent values.

This assumption can explain that the frequency of the oscillator 1 and that of the oscillator 2 are the same (Fig.4). Namely, the frequency of the oscillator 2 is equal to the frequency of the external inputs $(\alpha E_1,\ \beta E_1)$. Meanwhile, α and β control the amplitude of the oscillator 2.

When the values of α and β are in the specific range, for example $\alpha = 5.3$ and $\beta = 1.0$, the oscillator 2 oscillates aperiodically. This phenomenon would be due to the fact that the range of the external inputs to the oscillator is almost overlapped with oscillatory region (Fig.6 (b)). The mean amplitudes of $(\alpha E_1,\ \beta E_1)$ can be plotted in the oscillatory region. In this situation, the mechanism as "traveling over convergent states" in the case of periodic oscillation collapses. The oscillation is affected by the external signals and its internal oscillatory characteristics. Various lower frequencies emerge compared with periodic oscillation (Fig.5 (b)).

The results suggest that the periodic oscillation of the external input determines whether the output oscillation is periodic or aperiodic. Thus we examined if these phenomena are the case with a sinusoidal external input. $E_2(t)$ is shown in Fig.7 with external inputs $(P, Q) = (\alpha(0.12\sin 2\pi\omega t + A),\ \beta(0.12\sin 2\pi\omega t + A))$, where $\omega = 0.00365$ and $A = 0.23754$. These are adjusted so as to be equivalent to the inputs from oscillator 1 in the previous numerical experiments. Oscillatory behaviors in this case (Fig.7) are similar to those in Fig.4 and 5. Therefore, with at least periodic external inputs, periodic or aperiodic oscillation may be produced on a single Wilson-Cowan oscillator. And then, we change the frequency of the input sinusoidal function. The oscillator reveals different behaviour as the input frequency changes (Fig.8). Aperiodic oscillation occurs at only specific input frequencies which are close to that Wilson-Cowan oscillator has. Large frequencies lead to a state in which the oscillator behaves as if it has an average constant parameter value. On the other hand, small frequencies give rise to oscillation that is formed by two periodic waves. We guess that one wave has the frequency of Wilson-Cowan oscillator, and another wave has that of the periodic inputs.

3.4. Chaotic characteristics

It has been reported that chaotic oscillation is observed on reciprocal connected Wilson-Cowan oscillators at particular strengths of connections [8]. Thus it is also possible that induced aperiodic oscillation of the oscillator 2 is chaotic.

We calculated the largest Lyapunov exponent (LLE) of the oscillator 2 (Fig.9). When a perturbation was introduced to the initial state in the state-space, the mean rate of expansion corresponds to the largest Lyapunov exponent. The expansion rate was averaged over 50000 steps.

In the specific range of α and β ($1.13 \lesssim \alpha\overline{E}_1 \lesssim 1.37$

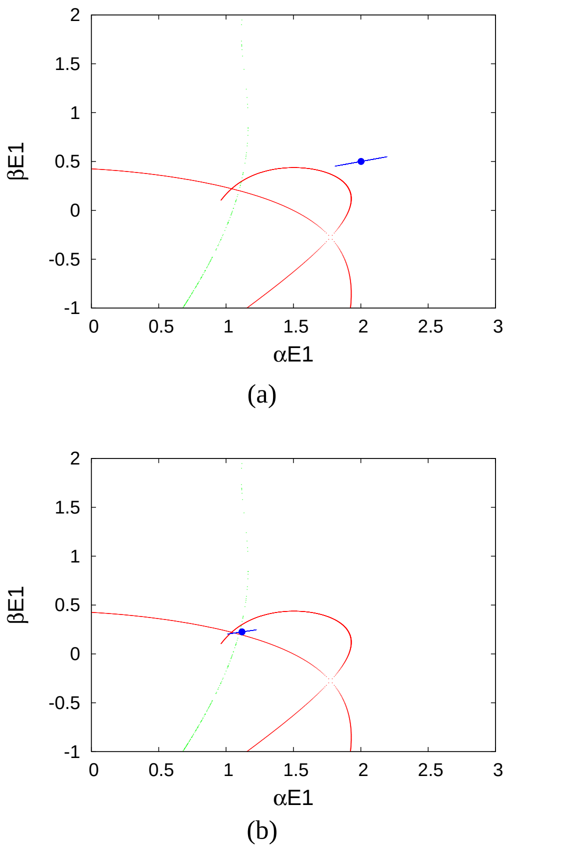


Figure 6: The range of the external inputs (blue line) in the bifurcation set space. The central point denotes the average values. (a) $\alpha = 8.00,\ \beta = 2.00$ (b) $\alpha = 4.47,\ \beta = 0.90$

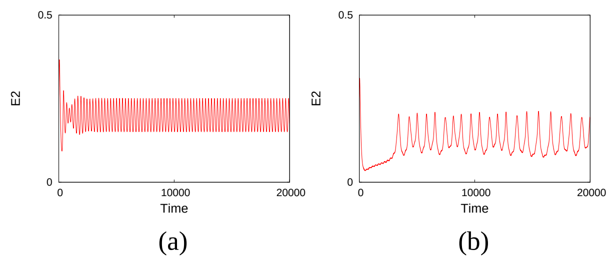


Figure 7: Periodic and aperiodic oscillation E_2 with sinusoidal external inputs. (a) $\alpha = 8.00,\ \beta = 2.00$. (b) $\alpha = 4.45,\ \beta = 0.75$.

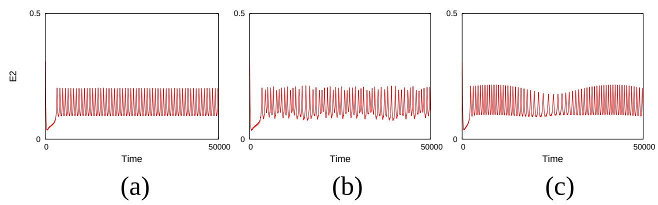
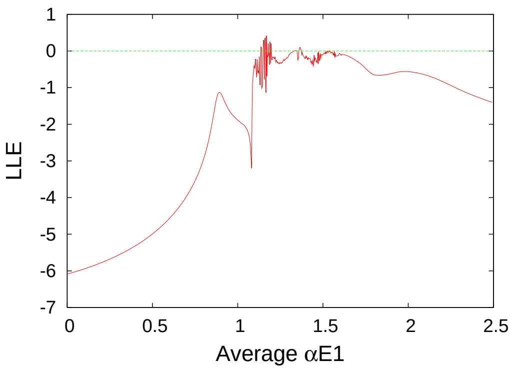


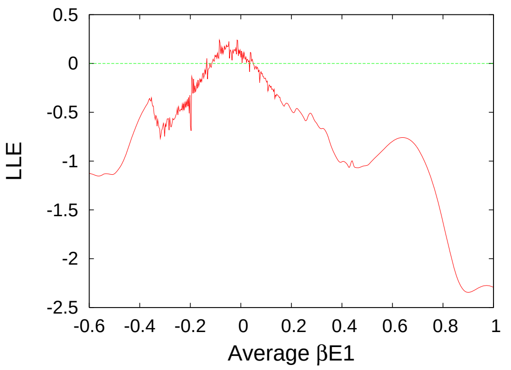
Figure 8: The oscillatory mode with sinusoidal inputs changes depending on a frequency of the inputs. (a) $\omega = 0.3$. (b) $\omega = 0.003891$. (c) $\omega = 0.00003$.

in Fig.9 (a), and $-0.11 \lesssim \beta\overline{E}_1 \lesssim 0.04$ in Fig.9 (b)), LLE can be positive, which implies chaotic instability. This result suggests that unidirectional connected Wilson-Cowan oscillators can also produce chaotic oscillations as the reciprocally connected model can.

As has been mentioned, aperiodic oscillation observed in this study can occur with sinusoidal external inputs. LLEs in this case are much similar to those in Fig.9. Therefore the oscillation in response to the sinusoidal input can also become chaotic.



(a)



(b)

Figure 9: Largest Lyapunov exponents. (a) $P = \alpha\overline{E}_1, Q = 0.1$. (b) $P = 1.6, Q = \beta\overline{E}_1$.

4. Conclusion

In this study, we investigated the properties of oscillations of unidirectionally connected Wilson-Cowan oscillators. Inputs from the oscillator oscillating periodically induces periodic or aperiodic oscillations dependent on the strengths of connections. The aperiodic oscillation is derived from the interaction between periodic inputs and the oscillator-owned oscillatory property. In addition, the frequency of input also affects the oscillation. Thus aperiodic oscillation can be obtained on a single Wilson-Cowan oscillator by periodic inputs. Moreover, the largest Lyapunov exponent suggests that its oscillation would be chaotic at a particular range of strengths of connections. Although this model is much simple, complex behaviours can be induced. Thus this may be useful as a tool to construct a complicated system such as a brain model in order to simulate complex oscillatory phenomena.

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