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Exponential Transient Phase Waves in a Ring of Unidirectionally Coupled Parametric Oscillators

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Abstract– Phase waves rotating in a ring of unidirectionally coupled parametric oscillators are studied. The system has a pair of spatially uniform stable periodic solutions with a phase difference as well as unstable quasiperiodic traveling phase wave solutions, which are generated from the origin through a period doubling bifurcation and the Neimark-Sacker bifurcation, respectively. In transient states, phase waves rotating in a ring are generated, the duration of which increases exponentially with the number of oscillators (exponential transients).

1. Introduction

In spatially extended bistable systems, it has been shown that transient spatially non-uniform patterns last for a long time, the duration of which increases exponentially with system size. Then systems never reach their asymptotic spatially uniform steady states in a practical time when system size is large, and transient states can not be discriminated from steady states. These exponential transients have been shown in a one-dimensional reaction-diffusion equation (the time-dependent Ginzburg-Landau equation), in which a kink (a front) or pairs of a kink and antikink (pulses) exists in transient states [1 – 3]. Similar spatial patterns have been shown in various reaction-diffusion systems and convection-diffusion systems, which are referred to as metastable dynamics [4, 5]. Recently, the authors have shown exponential transient oscillations which are rotating waves in a ring of unidirectionally coupled sigmoidal neurons [6] as well as in bistable rings of coupled maps [7, 8]. Since discrete-time map systems are regarded as the Poincaré maps in continuous-time systems, it is expected that exponential transient states exist in systems which have bistable periodic solutions.

In this study, we consider a ring of unidirectionally coupled parametric oscillators and phase waves rotating in it. A single oscillator shows bistable parametric oscillations with a phase difference π [9]. A ring of oscillators then has bistable spatially uniform periodic solutions. In addition, it has unstable quasiperiodic traveling wave solutions in which the phases of the oscillations of elements rotate in a ring. These rotating phase waves last for exponentially long time until the system reaches one of stable spatially uniform oscillations.

2. Unidirectionally Coupled Parametric Oscillators

We consider the following coupled oscillators.

$$\begin{aligned} &\mathrm{d}u_n(t)/\mathrm{d}t = v_n(t) \\ &\mathrm{d}v_n(t)/\mathrm{d}t = -u_n(t) - \gamma v_n(t) - \beta u_n(t) v_n^{\,2}(t) \\ &\qquad + \alpha \sin(2t) u_n(t) + c(u_{n-1}(t) - u_n(t)) \\ &(1 \le n \le N, \quad u_0(t) = u_N(t)). \end{aligned} \tag{1}$$

A total N of parametric oscillators [9] are unidirectionally coupled with strength c (≥ 0) and make a closed loop with $u_0 = u_N$. A coefficient $\alpha\sin(2t)$ of u_n in the right-hand side of the second equation varies sinusoidally with a period π. The values of parameters are fixed as β = 1.0, γ = 0.5 and c = 0.2, and then α is used for a bifurcation parameter. The Poincaré map of ($u_1(t)$, $v_1(t)$) at $t = m\pi$ (m: integer) is used for bifurcation analysis of Eq. (1).

First, Fig. 1(a) shows a bifurcation diagram of a single oscillator (N = 1). We omit the element numbers for a single oscillator. The origin ($u = v = 0$) is a stable steady state for α < 1.014. It is destabilized through a subcritical period doubling bifurcation at α ≈ 1.014 (PD) and a pair of unstable solutions of period two is generated. A pair of stable solutions of period two with the large amplitude also exists and two pairs of the solutions disappear through saddle-node bifurcations at α ≈ 0.961 (SN). The stable pair corresponds to parametric oscillations with a period 2π and a phase difference between them is π. Figure 1(b) shows the trajectory of a stable periodic solution for α = 1.1 obtained with computer simulation of Eq.(1), in which open circles (○) denote the states at $t = m\pi$. Further, they become chaotic at α ≈ 1.8 through a pitchfork bifurcation at α ≈ 1.681 (PF) and a cascade of period doubling bifurcations over α ≈ 1.793.

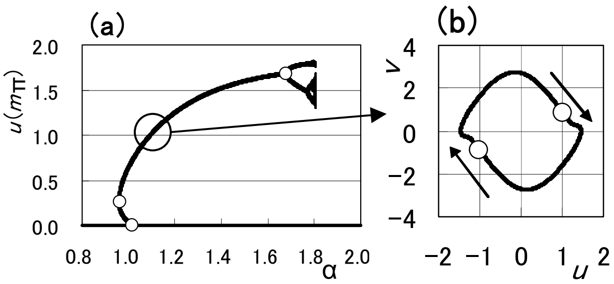

Fig. 1. Bifurcation diagram (a) and trajectory (b) of a single oscillator.

Next, Fig. 2 shows a bifurcation diagram of Eq. (1) with six oscillators ($N = 6$) about $\alpha \approx 1$. As α increases, a stable invariant closed curve is generated from the origin ($(u_n, v_n) = (0, 0)$ ($1 \le n \le N$)) through the Neimark-Sacker bifurcation at $\alpha \approx 0.988$ (NS1). The generated quasiperiodic solutions on it are traveling phase waves rotating in the ring because of the invariance of Eq. (1) under cyclic shifts in n. Further, they lie in the subspace: $u_{n+N/2} = -u_n$, $v_{n+N/2} = -v_n$ ($1 \le n \le N/2$) when N is even since Eq. (1) is invariant under changes in the signs of u_n and v_n. Although the branch of the unstable closed invariant curve can not be traced with our method, traveling phase waves are observed with computer simulation putting initial conditions in the invariant subspace, e.g.

$$\begin{aligned} u_n(0) &= v_n(0) = -1.0 \quad (1 \le n \le l_h) \\ u_n(0) &= v_n(0) = 1.0 \quad (l_h < n \le N) \qquad (l_h = N/2). \end{aligned} \tag{2}$$

They are observed stably in the subspace when $\alpha > 0.984$. Hence it is expected that a pair of closed invariant curves is generated through a saddle-node bifurcation at $\alpha \approx 0.984$ (SN1) and the unstable branch is connected to the origin at $\alpha \approx 0.988$ through some bifurcations. (Maximum values of $u_1(t)$ are plotted for the quasiperiodic traveling phase waves in Fig.2.)

On the other hand, pairs of solutions of period two are generated through saddle-node bifurcations at $\alpha \approx 0.961$ (SN) in the same manner as a single oscillator. The stable pair corresponds to spatially uniform parametric oscillations with a period 2π and a phase difference π. The unstable pair causes the Neimark-Sacker bifurcation at $\alpha \approx 0.971$ (NS) and connects to the origin through a period doubling bifurcation at $\alpha \approx 1.014$ (PD).

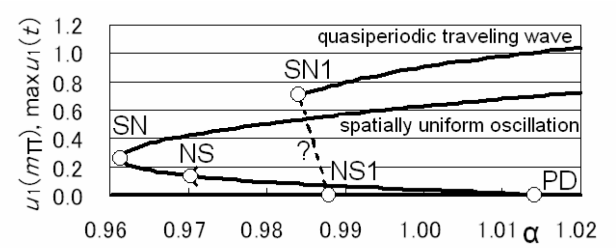

Fig. 2. Bifurcation diagram of a ring of six oscillators.

Figure 3 shows an example of the traveling phase waves obtained with computer simulation of Eq. (1) with $N = 6$, $\alpha = 1.1$ under the initial condition Eq. (2). Plotted are time courses of $v_n(t)$ ($1 \le n \le N$) (a) and snapshots of spatial patterns $v_n(t)$ at $t = 2m\pi$ (m = 664, 672, 680) (b). The amplitude and phase of each oscillator changes quasiperiodically, which rotates in the ring. Further, Fig. 4 shows the Poincaré map of $(u_1(t), v_1(t))$ at $t = 2m\pi$ (a) and sequences $u_1(2m\pi)$ and $v_1(2m\pi)$ (b) ($1001 \le m \le 1100$). The state jumps and rotates clockwise on an invariant curve (Successive 20 points are plotted with open circles in (a)), which corresponds to changes in the phase of the first oscillator.

It is shown with computer simulation of Eq. (1) that the traveling phase waves are unstable except for α close to the bifurcation point at the origin ($\alpha \approx 0.988$). The system reaches one of the stable spatially uniform oscillations when $\alpha > 0.988$ eventually, but rotating phase waves appear generally in transient states. Figure 5 shows an example of transient phase waves in a ring of ten oscillators. It rotates four times in a ring and converges to a spatially uniform oscillation at $t \approx 770$. It will be shown in Sect. 4 that the duration of such transient phase waves increases exponentially with the number of oscillators.

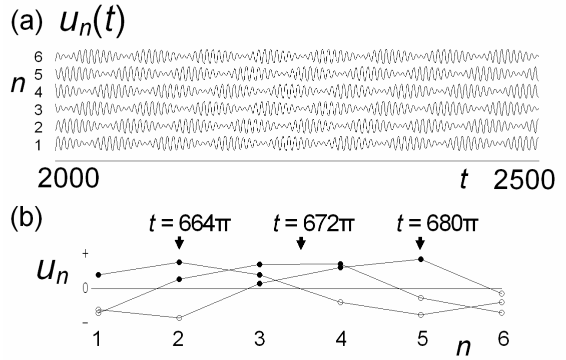

Fig. 3. Traveling phase wave in a ring of six oscillators.

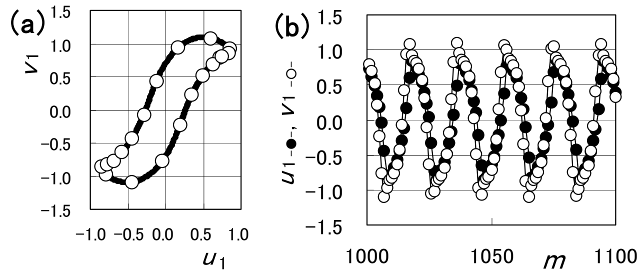

Fig. 4. Traveling phase wave at $t = 2m\pi$.

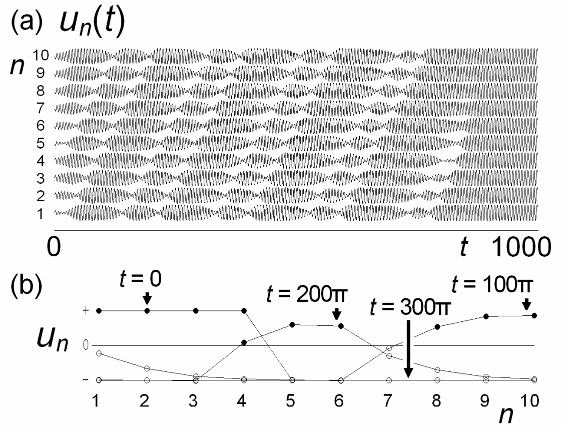

Fig. 5. Transient phase wave in a ring of ten oscillators.

3. A Coupled Map Model

In transients of a single oscillator to one of stable spatially uniform oscillations, the states quickly approach a one-dimensional curve. Figure 6(a) shows the Poincaré map of $(u(t), v(t))$ at $t = 2m\pi$ (m: integer) in transient states of a single oscillator with α = 1.1. Plotted are ($u(2m\pi)$, $v(2m\pi)$) ($11 \le m \le 20$) obtained with 1000 runs of computer simulation under random initial conditions: $u(0)$, $v(0) \sim U(-1.5, 1.5)$. The states approach the fixed points (u_f, v_f) ≈ (±1.01, ±0.87) (open circles) along a one-dimensional curve (solid circles) as two examples plotted with open triangles and open squares, and hence the approach is expressed with a one-dimensional map.

Figure 6(b) then shows a return map $u(2(m+1)\pi) = g(u(2m\pi))$ for α = 1.1 (a solid line). It is approximated by a sinusoidal function of u (crosses)

$$g(u) = u + A\sin(\pi u / u_f) \quad (A = 0.22, \quad u_f = 1.01). \quad (3)$$

Equation (1) is thus approximated by a discrete-time system with $g(u)$ as

$$u_n(\tau+1) = g(u_n(\tau)) + c'(u_{n-1}(\tau) - u_n(\tau)) \\ (\tau = t/(2\pi), \quad 1 \le n \le N, \quad u_0(\tau) = u_N(\tau)). \quad (4)$$

where unidirectional linear coupling is used and c' corresponds to coupling strength.

The bifurcations of spatially uniform states ($u_n(t) = u(t)$ ($1 \le n \le N$)) in Eq. (4) are the same as those of a single oscillator (N = 1). Figure 7 shows differences $g(u) - u$ for α = 0.95, 0.96, 0.97, 1.0 and 1.02. The origin (u = 0) is a globally stable fixed point when $\alpha \le 0.95$. As α increases, pairs of stable and unstable fixed points are generated through saddle-node bifurcations at $\alpha \approx 0.96$. A pair of generated stable fixed points corresponds to stable parametric oscillations with a period 2π and a phase difference π in Eq. (1). A pair of generated unstable fixed points then disappears and the origin is destabilized through a subcritical pitchfork bifurcation at $\alpha \approx 1.01$.

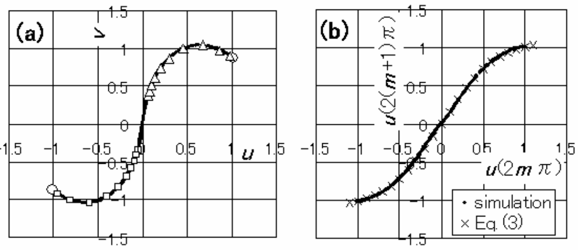

Fig. 6. Poincaré map (a) and a return map (b) of a single oscillator.

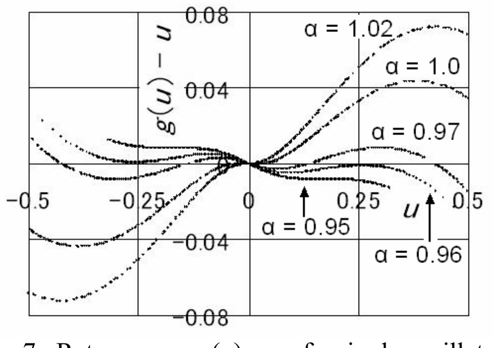

Fig. 7. Return maps $g(u) - u$ of a single oscillator.

A further increase in α causes the Neimark-Sacker bifurcations at the origin in Eq. (4) and spatially nonuniform solutions are generated. That is, the eigenvalues of the Jacobian matrix evaluated at the origin ($u_n = 0$ ($1 \le n \le N$)) is given by

$$\lambda_n = g'(0) - c' + c'\exp(j2n\pi / N) \quad (0 \le n \le N-1). \quad (5)$$

As α increases, $g'(0)$ increases and the absolute values of pairs of complex conjugate eigenvalues cross unity successively. Then closed invariant curves and unstable quasiperiodic solutions on them are generated. The quasiperiodic solutions are traveling waves rotating in the ring because of the invariance of Eq. (4) under cyclic shifts in n. The first ones lie in the subspace: $u_{n+N/2} = -u_n$ ($1 \le n \le N/2$) when N is even since $g(u)$ is antisymmetric. They correspond to the traveling phase waves in Eq. (1) and one-dimensionally unstable if the bifurcation is supercritical. Hence, the same two kinds of bifurcations (a pitchfork and the Neimark-Sacker bifurcations) as Eq. (1) occur at the origin in Eq. (4), while the order of the appearance of them is reversed.

When α = 1.1 with $g(u)$ in Eq. (3), the absolute values of the first complex conjugate eigenvalues of the Jacobian matrix are larger than unity since $g'(0) = 1 + A\pi/u_f \approx 1.68$ and the Neimark-Sacker bifurcation has occurred. Although the type of the bifurcation (super or sub-critical) is unclear, the closed invariant curve and quasiperiodic solutions on it are observed with computer simulation. The wave forms of the quasiperiodic solutions obtained with computer simulation of Eq. (4) with Eq. (3) under the symmetric initial condition $u_n(0)$ in Eq. (2) are similar to $u_1(2m\pi)$ in Fig. 4(b). Equation (4) is bistable and rotating waves are generated in transient states as Eq. (1). In the following, it will be shown that the kinematics and duration of rotating waves in Eq. (4) agree with those in Eq. (1).

4. Exponential Duration of Transient Phase Waves

4.1. Kinematics of Pulse Fronts of Phase Waves

Considering spatial patterns $u_n(t)$ at $t = 2m\pi$ of the traveling phase waves as shown in Fig. 3(c), the signs of a half of oscillators are positive and those of the other half are negative. We refer to each half as a pulse and boundaries between pulses as pulse fronts. Two pulse fronts propagate at the same speeds.

Figure 8(a) shows the speed v (the number of oscillators in which pulse fronts propagate per time t) of the traveling phase wave against the pulse width l_h (a half of the number of oscillators (= $N/2$)), which were obtained with computer simulation of Eqs. (1) and (4) under a symmetric initial condition as Eq. (2). A value of the coupling strength c' in Eq. (4) was taken to be 0.4398 so that the speed $v(l_h \to \infty)$ agrees with that of Eq. (1). The speed increases with pulse width and approaches the values in the limit of $l_h \to \infty$. Further, Fig. 8(b) shows a semi-log plot of a difference between $v(\infty)$ and $v(l_h)$ against the pulse width l_h ($v(30)$ is used for $v(\infty)$). The difference decreases exponentially with the pulse width and is extremely small when the number of oscillators is large.

Using this exponential dependence of the speeds of the traveling phase waves on the pulse width, changes in the locations l_1 and l_2 of pulse fronts and the pulse width $l = l_2 - l_1$ in asymmetric rotating phase waves are described by

$$\begin{aligned} &\mathrm{d}l_1/\mathrm{d}t = v(N-l), \quad \mathrm{d}l_2/\mathrm{d}t = v(l) \\ &\mathrm{d}l/\mathrm{d}t = v(N) - v(N-l) \\ &\quad v(l) = v(\infty) - k\exp(-rl) \\ &\quad (1 \le l_1 \le l_2 \le N), \end{aligned} \tag{6}$$

where values of k and r are estimated from the graphs in Fig. 8(b) as $k = 0.15$ and $r = 1.18$ for Eq. (1) and $k = 0.14$ and $r = 1.39$ for Eq. (4). Here we assume that the propagation speeds of pulse fronts depend on their backward pulse width as [7, 8] not on their forward pulse width as [6] since $\mathrm{d}v/\mathrm{d}l_h > 0$. This dependency still needs further investigation, but the duration of the phase waves are well explained with Eq. (6) as shown below.

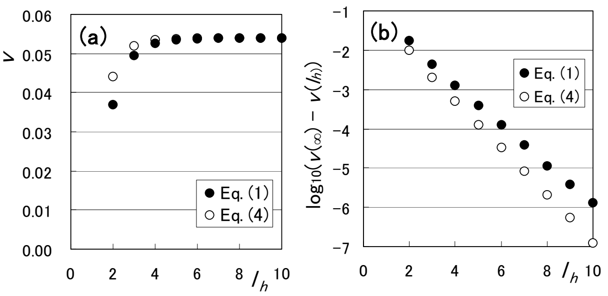

Fig. 8. Speeds of traveling phase waves.

4.2. Duration of Asymmetric Phase Waves

Replacing l_h in the initial condition Eq. (2) with l_0 ($1 \le l_0 < N/2$), transient asymmetric rotating phase waves are generated in Eq. (1). The duration T of them is derived with Eq. (6) letting $l(0) = l_0$ and $l(T) = 0$ as

$$\begin{aligned} T(l_0; N) = \frac{\exp(rN/2)}{kr}\{&\mathrm{arctanh}[\exp(r(l_0 - N/2))] \\ &- \mathrm{arctanh}[\exp(-rN/2)]\}. \end{aligned} \tag{7}$$

By letting N be infinity, simpler forms of Eqs. (6) and (7) are given by

$$\begin{aligned} &\mathrm{d}l/\mathrm{d}t = -k\exp(-rl) \\ &l(t) = 1/r \cdot \log[\exp(rl_0) - krt] \qquad (l(0) = l_0) \\ &T(l_0) = [\exp(rl_0) - 1]/(kr) \qquad (l(T) = 0). \end{aligned} \tag{8}$$

The duration of rotating phase waves thus increases exponentially with the initial pulse width.

Figure 9 shows a semi-log plot of the duration T of phase waves against the initial pulse width l_0 in a ring of 21 oscillators ($N = 21$). Plotted are the results of computer simulation of Eq. (1) under Eq. (2) with l_0 ($1 \le l_0 < N/2$) instead of l_h (closed circles), those of Eq. (4) with Eq. (3) (open circles) and Eq. (8) (lines). Although the results of the coupled map model (Eq. (4)) slightly differ from those of the original system (Eq. (1)), Equation (8) agrees with both simulation results.

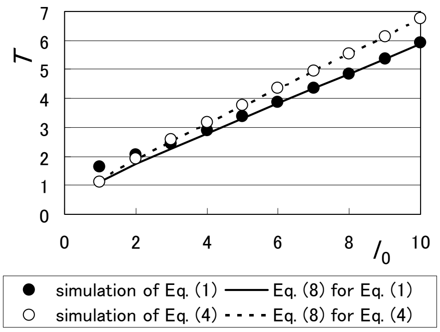

Fig. 9. Duration of phase waves vs initial pulse width l_0.

5. Conclusion

Exponential transient phase waves in a ring of unidirectionally coupled parametric oscillators were shown. A coupled map model for them was also derived. The propagation and duration of phase waves were well described by a kinematical equation. This is the first report on exponential transient phase waves in coupled oscillators as far as the authors know. Since the coupling term used in this study is ad-hoc and artificial, however, a more physical model will be proposed and examined.

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