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Pattern formation of self-organized feature extraction in a three-dimensional discrete reaction-diffusion system

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Abstract—We investigated pattern formations in a three-dimensional discrete FitzHugh-Nagumo model. We found self-organization of the edge planes structure of an initial cubic pattern in three dimensions. As same as in two-dimensional model, self-organization of the edge points and the edge lines were also developed spontaneously in the three-dimensional model. These pattern formations can be regarded as a function of feature extraction self-organized in the three-dimensional discrete reaction-diffusion system.

1. Introduction

Spatial/temporal order is sometimes self-organized in an open system far from equilibrium. Pattern formation on the skin surface of some kinds of animal and chemical waves in Belousov-Zhabotinsky (BZ) reaction are one of examples of them. It is interesting that a phenomenon on photosensitive BZ reaction generates the contour and/or the reversal pattern of an input light pattern [1]. For simulating formation of these patterns, reaction-diffusion models are often adopted [2]. Many numerical simulations of the pattern formation were restricted to one or two spatial dimensions. However, several particular structures of three dimensions have been reported recently [3, 4]. It seems important to study three-dimensional self-organized structures because we live in three-dimensional worlds.

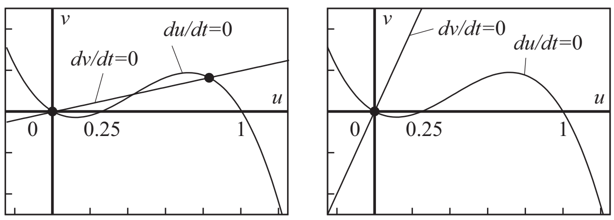
The above pattern formation induced by photosensitive BZ reaction brings the contour and/or the reversal structure of the input pattern like image processing. However, these specific patterns do not remain stationary at all. On the other hand, Nomura et al. [5] reported that a FitzHugh-Nagumo model brought stationary patterns like typical image processing. The functions of the pattern formation are extracting edge points, edge lines, and image segmentation in numerical simulations. Ebihara et al. [6] also showed that spatial discreteness or small value of diffusion coefficients was important factor for obtaining those stationary patterns. Our main interest in this study is to clarify the behavior of the discrete reaction-diffusion model in three dimensions. On discrete reaction-diffusion models, there are few researches dealing with three-dimensional pattern formation. Throughout this study, we regard the term “discrete system” as a system that has small spatial connection between neighboring cells: a few diffusion or large spatial interval.

The purpose of this study is to investigate pattern formations of the discrete reaction-diffusion model in three dimensions. We confirmed that the model extracted the edge points and the edge lines of an initial cubic pattern in a self-organized fashion. And we found that the edge planes structure of the pattern was organized spontaneously. The edge planes structure appears only in three-dimensional model.

2. Related researches

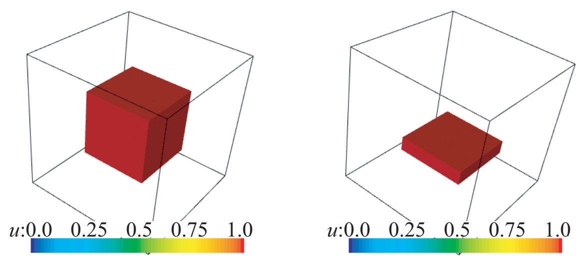
2.1. Pattern formation in three dimensions

Leppänen et al. [4] discussed on comparison between pattern formations in three dimensions and in two dimensions. They investigated a general Turing system and the Gray-Scott model in their numerical simulations. Ohta and



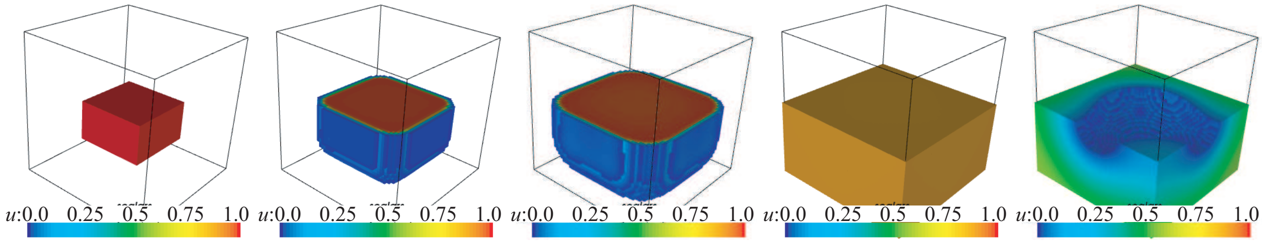
(a) (b)

Figure 1: Nullclines of the FitzHugh-Nagumo system of bistable (b = 10.0) (a) and monostable (b = 1.0) (b). The filled circles mean stable points. a = 0.25 in both cases.



(a) (b)

Figure 2: A given shape for initial condition of the system. A distribution of u shaped solid cube and its value was u = 1.0 (a). (b) represents a sliced image of (a).



(a) (b) (c) (d) (e)

Figure 3: A wave propagation in three-dimensional FitzHugh-Nagumo model. A half of lower part of distributions of u at 0th step (a), 50th step (b), 100th step (c), 500th step (d), and 940th step (e) are shown. $D_u = 4.0 \times 10^{-2}$, $D_v = 1.6 \times 10^{-1}$, $a = 0.1$, $b = 1.0$, $\varepsilon = 1.0 \times 10^{-4}$, $\Delta x = 0.01$, $\Delta t = 1.0 \times 10^{-4}$.

co-workers studied the pattern formation in three dimensions for a type of FitzHugh-Nagumo model, the Brusselator, and the Gray-Scott model in numerical simulations [3]. They searched various parameters and reported several categories of three-dimensional structure. However, they did not mention a discreteness of the system, e.g., no parameter searching for various values of diffusion coefficients or spatial intervals.

2.2. Image processing like behavior by the FitzHugh-Nagumo model

As mentioned above, Nomura et al. [5] reported that the FitzHugh-Nagumo model (Eqs. (1) and (2)) behaved as self-organized image processing, including edge points/lines detection and image segmentation.

$$\frac{\partial u}{\partial t} = D_u \nabla^2 u + \frac{1}{\varepsilon}\{u(u-a)(1-u) - v\}, \quad (1)$$

$$\frac{\partial v}{\partial t} = D_v \nabla^2 v + u - bv, \quad (2)$$

where activator u and inhibitor v are variables. D_u and D_v are diffusion coefficients of u and v, respectively. a, b and ε are constant parameters. Image processing like behaviors mentioned above appeared when the system was discrete [6], i.e., the spatial interval Δx was moderately large or the diffusion coefficients D_u and D_v were moderately small.

Figure 1 shows nullclines of the Eqs. (1) and (2) when D_u and D_v equal to zero. As the model is under bistable system (see Fig. 1(a)), this system yields segmented image as a binarization of the input pattern This is easy to understand because the value of each element of the model settle down at either stable point in the bistable system. While, as the model is under monostable system (Fig. 1(b)), this system yields the edge points or the edge lines of the input pattern. It seemed that the diffusion coefficients D_u and D_v affect the number of stable points of the system, and the system turns bistable only on the edge regions (see detailed discussion in subsection 4.2).

3. Pattern formation in three-dimensional FitzHugh-Nagumo model

For computer simulation, we adopted the explicit Euler method on the FitzHugh-Nagumo equation (Eqs. (1) and (2)). The cell size was Δx and the number of the cell was $N = 50$, and temporal interval was represented as Δt. There were no flux at all the boundaries of the system; so-called Neumann condition was adopted. The initial condition shaped a solid cube (see Fig. 2). We carried on a numerical simulation varying the diffusion coefficients D_u and D_v, because our purpose is to investigate relationship between discreteness of the system and three-dimensional pattern formation.

When the diffusion coefficients were sufficiently large ($D_u = 4.0 \times 10^{-2}$ and $D_v = 1.6 \times 10^{-1}$), three-dimensional wave propagation appeared as expected (see Fig. 3). On the other hand, Fig. 4 represents a result when D_u was the smallest value ($D_u = 4.0 \times 10^{-5}$) in current numerical simulations. The edge points of the given shape for initial condition appeared as a convergent pattern in this case. When D_u was 5.0×10^{-5}, the edge lines appeared (Fig. 5). These results were similar to the results in two dimensions [5]. Moreover, as shown in Fig. 6, we obtained the edge planes structure of the initial shape when D_u was 1.5×10^{-4}. The static edge planes structure can be regarded as a characteristic pattern self-organized in three-dimensional model.

Additionally, we carried out a numerical simulation for sphere shaped initial condition and obtained the edge plane on the same parameter values for the cube shape.

4. Discussions

4.1. Pattern formation

We found a function of self-organized feature extraction that appeared in three-dimensional FitzHugh-Nagumo model. The function includes extraction of the edge planes of the initial shape in three-dimensonal space. Other feature extractions (on edge points/lines detection) were also confirmed in three dimensions as well in two dimensions when the diffusion coefficients D_u and D_v were moderately small. When D_u and D_v were too small, the value of all the cells settle down at zero.

In current numerical simulations, we observed the three-dimensional wave propagation, the edge points structure, the edge lines structure, and the edge planes structure. We also observed so-called Turing pattern in $N = 100$ (see Fig. 7), however we could not confirm the pattern clearly

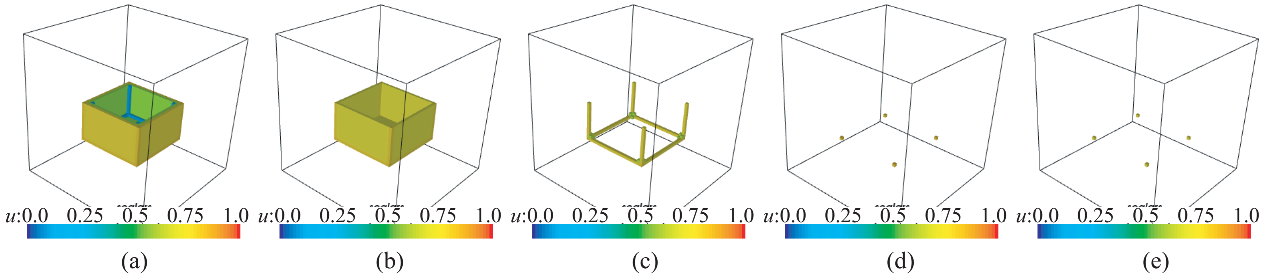


Figure 4: A time evolution of the edge points structure observed in the discrete system with $D_u = 4.0 \times 10^{-5}$, $D_v = 1.6 \times 10^{-4}$. A half of lower part of distributions of u at 1600th step (a), 1700th step (b), 1900th step (c), 2500th step (d), and 5000th step (e) are visualized. The other parameters are $a = 0.1$, $b = 1.0$, $\varepsilon = 1.0 \times 10^{-4}$, $\Delta x = 0.01$, $\Delta t = 1.0 \times 10^{-4}$.

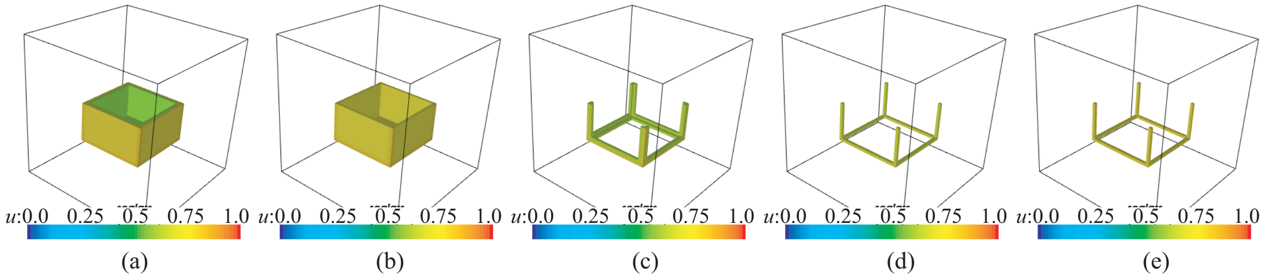


Figure 5: A time evolution of the edge lines structure observed in the discrete system with $D_u = 5.0\times10^{-5}$, $D_v = 2.0\times10^{-4}$. A half of lower part of distributions of u at 1600th step (a), 1700th step (b), 1900th step (c), 2500th step (d), and 5000th step (e) are visualized. The other parameters are $a = 0.1$, $b = 1.0$, $\varepsilon = 1.0 \times 10^{-4}$, $\Delta x = 0.01$, $\Delta t = 1.0 \times 10^{-4}$.

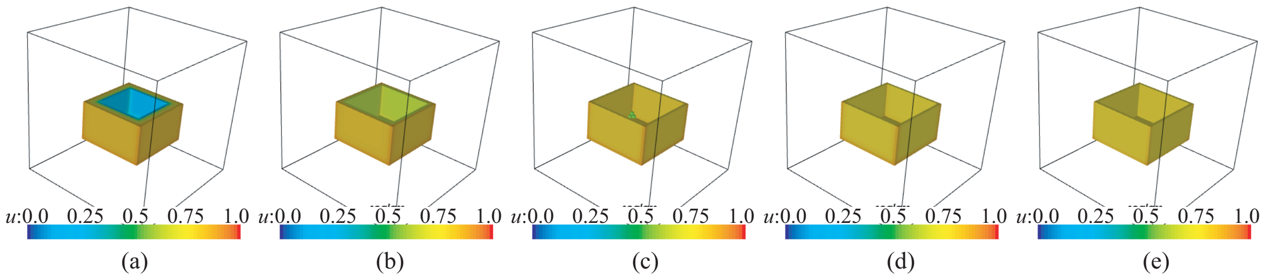


Figure 6: A time evolution of the edge planes structure found in $D_u = 1.5\times10^{-4}$ and $D_v = 6.0\times10^{-4}$. A half of lower part of distributions of u at 1600th step (a), 1700th step (b), 1900th step (c), 2200th step (d), and 5000th step (e) are visualized. The other parameters were $a = 0.1$, $b = 1.0$, $\varepsilon = 1.0 \times 10^{-4}$, $\Delta x = 0.01$, $\Delta t = 1.0 \times 10^{-4}$.

in $N = 50$. Various parameters searching are needed in the proposed FitzHugh-Nagumo model to observe more various Turing patterns.

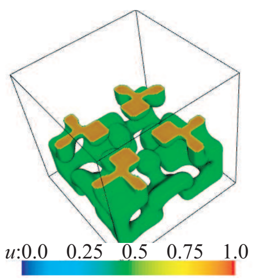


Figure 7: A three-dimensional Turing pattern found in $a = 0.2$, $b = 5.0$, $\varepsilon = 1.0 \times 10^{-3}$, $D_u = 1.0 \times 10^{-2}$, $D_v = 4.0\times10^{-2}$, $\Delta x = 0.01$, and $\Delta t = 1.0\times10^{-4}$. A half of lower part of the isosurface $u = 0.5$ at 300000th step is visualized.

4.2. A brief theory of extracting the edge structures in the FitzHugh-Nagumo model

Here, we consider the FitzHugh-Nagumo system in one spatial dimension, for better understanding of the discussion. It seems not to lose the generality of the theory for adopting it in three dimensions. In addition, we assume that D_u is zero. We have confirmed that the edge structures appear also in such case. First, we adopt a central difference in space and obtain the following Eqs. (3) and (4);

$$\frac{du_i}{dt} = \frac{1}{\varepsilon}\{u_i(u_i - a)(1 - u_i) - v_i\}, \tag{3}$$

$$\frac{dv_i}{dt} = r'(v_{i+1} - 2v_i + v_{i-1}) + u_i - bv_i, \tag{4}$$

where r' represents $D_v/(\Delta x)^2$. The subscript $i = 1, 2, ..., N$ is a spatial index. Equations (5) and (6) represent nullclines of Eqs. (3) and (4), respectively (see also Fig. 8);

$$v_i = u_i(u_i - a)(1 - u_i), \tag{5}$$

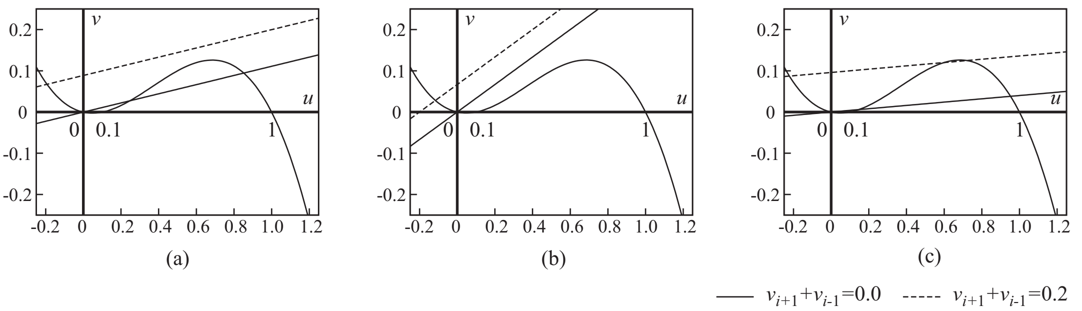


Figure 8: Nullclines of Eqs. (3) and (4) at $r' = 4.0$ (a), $r' = 1.0$ (b), and $r' = 12.0$ (c). $\Delta x = 0.01$ and $a = 0.1$, $b = 1.0$ in each case. The solid lines represent nullclines when $v_{i+1} + v_{i-1}$ is small (0.0) and the dashed lines show that when $v_{i+1} + v_{i-1}$ is sufficiently large (0.2). Efficiently large discreteness results in non-uniform number of stable point in space followed by a distribution of the inhibitor value v (a). In the other cases, numbers of stable point are identically in space (b),(c).

$$v_i = \frac{u_i}{b + 2r'} + \frac{r'(v_{i+1} + v_{i-1})}{b + 2r'}, \qquad (6)$$

where b, r', v_{i+1}, and v_{i-1} are positive.

When D_v is moderately small or Δx is moderately large (i.e., r' is moderately small), a number of the stable point varies spatially followed by the values of neighboring inhibitors v_{i+1} and v_{i-1}. The system has two stable points where the term $v_{i+1} + v_{i-1}$ is small, whereas the system has only one stable point around the origin where value of $v_{i+1} + v_{i-1}$ is efficiently large, because the line drawn by Eq. (6) moves to too upper region (see Fig. 8(a)). On one hand, when r' is too small, all spatial components of the system have one stable point as shown in Fig. 8(b). On the other hand, when r' is too large, all components of the system have two stable points as shown in Fig. 8(c). The sum of neighboring cells' inhibitor values is smaller in the edge cells than in the other cells. The grounds is that outer-side cells of the edge have almost zero inhibitor values v, since the inhibitor value almost depends on the activator value u. Thus, the model extracts the edge structures under moderate discreteness. In three dimensions, total sum value of neighboring inhibitor is larger than that in one dimension. Indeed, it becomes more complex when the D_u is not to zero. In such cases the cubic curve drawn by Eq. (5) also changes its profile.

5. Conclusion

In this study, we reported the research on the pattern formation in three spatial dimensions for the FitzHugh-Nagumo model. We confirmed that the proposed model self-organized the edge planes structure and the edge points/lines structure in the discrete system; the diffusion coefficients were moderately small. The former appeared as one of characteristic structure in the three-dimensional discrete reaction-diffusion system. The latter is common structure in one and/or two dimensions. As the diffusion coefficients increased, extracted pattern was changed the edge points, the edge lines, and the edge planes in order.

For sphere shaped initial condition, we also confirmed to obtain the edge plane structure. For more general shaped initial condition, we are required more effort and more computing power. According to a previous work [7], the model in two dimensions extracted the edge lines for arbitrary figures, e.g., a photograph of building. We consider that the edge detection for three-dimensional arbitrary object is possible because of an analogy of two-dimensional result. Nevertheless, we must confirm to obtain the edge planes for any arbitrary object in three dimensions. If the proposed model extracts it, the field of three-dimensional visualization, e.g., visualization of computed tomography might be one of application of this work.

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