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Some chaotic properties of the β-hysteresis transformation

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Abstract—A two-valued piecewise-linear, constant slope, 1D transformation is defined as a simplified model for hysteresis. By using Góra's S-matrix and classical results on Rényi's β-transformation $x \mapsto \beta x \pmod 1$ the density function of the induced absolutely continuous ergodic measure is determined. Interesting simulation results reveal the chaos established.

1. Introduction and the main result

The recent discovery of the memristor by Strukov et al. [8] calls the attention to investigating the qualitative properties of two-valued interval maps. We report on ongoing work explaining chaos in the piecewise expanding case. Our approach is based on observing that, *given a two-valued interval map $x \mapsto X(x)$ (satisfying certain conditions), a concatenated arclength parametrization leads to a single-valued interval map $s \mapsto S(s)$ for which standard results on chaos in 1D directly apply*. The general discussion in Section 3 involves two increasing branches. The definition of the β-hysteresis transformation, the simplest possible example is left to Section 4. Decreasing branches are considered only in two-valued Poincaré sections of the 2D Feng–Loparo ODE model example [3] for hysteresis in Section 2.

In line with the monograph of Boyarski & Góra [2] and the consecutive papers by Góra [4, 5], we focus our attention to absolutely continuous invariant measures for piecewise expanding 1D transformations, i.e., for interval maps of the form $\tau\colon [0,1] \to [0,1]$ satisfying $\tau|_{[a_n,a_{n+1}]} \in C^1$ for some finite partition $\cup_{n=0}^{N}[a_n, a_{n+1}] = [0,1]$ with $0 = a_0 < a_1 < \cdots < a_{N+1} = 1$ and $|\tau'(x)| \ge q > 1$ whenever $x \in [a_n, a_{n+1}]$, $n = 0, 1, \ldots, N$. (In general, τ is two-valued on the set $\{a_1, a_2, \ldots, a_N\}$.)

2. A two-valued, simple interval map modelling hysteresis

The piecewise linear 2D pair of ordinary differential equations

$$\begin{pmatrix}\dot x\\ \dot y\end{pmatrix} = \begin{pmatrix}\sigma & \omega\\ -\omega & \sigma\end{pmatrix}\begin{pmatrix}x+1\\ y\end{pmatrix} \quad \text{whenever } x \le 1 \qquad (1)$$

$$\begin{pmatrix}\dot x\\ \dot y\end{pmatrix} = \begin{pmatrix}\sigma & \omega\\ -\omega & \sigma\end{pmatrix}\begin{pmatrix}x-1\\ y\end{pmatrix} \quad \text{whenever } x \ge -1 \qquad (2)$$

is a well-known model for hysteresis [3, 6]. Here σ and ω are parameters satisfying, for simplicity, $\sigma \ge 0$ and $\omega > 0$. Solutions to system (1)–(2) can be started at points where the right-hand side is single-valued, i.e., at initial values $(x(0), y(0)) = (x_0, y_0) \in \mathbb{R}^2$ satisfying $x_0 < -1$ or $x_0 > 1$. Any time a trajectory governed by equation (1) reaches the right-hand switching line $x = 1$, it remains in the half-plane $\{(x,y) \in \mathbb{R}^2 \mid x \ge -1\}$, and it is governed by equation (2) for a while. Similarly, any time a trajectory governed by equation (2) reaches the left-hand switching line $x = -1$, it remains in the half-plane $\{(x,y) \in \mathbb{R}^2 \mid x \le 1\}$, and it is governed by equation (1) for a while (until it hits the switching line $x = 1$ etc.). Without specifying the respective domains,

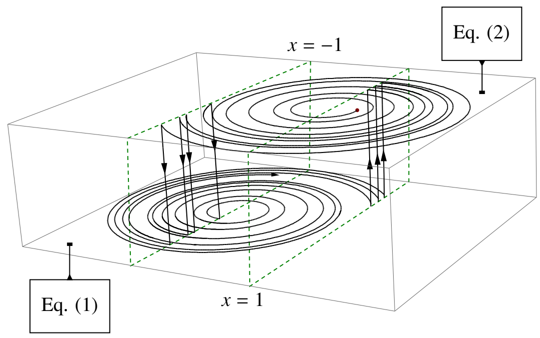

Figure 1: A typical trajectory of the system portrayed according to equations (1) and (2)

the solutions to system (1)–(2) can be written as

$$\begin{pmatrix}x(t,x_0,y_0)\\ y(t,x_0,y_0)\end{pmatrix} = \begin{pmatrix}e^{\sigma t}\cos(\omega t) & e^{\sigma t}\sin(\omega t)\\ -e^{\sigma t}\sin(\omega t) & e^{\sigma t}\cos(\omega t)\end{pmatrix}\begin{pmatrix}x_0 \mp 1\\ y_0\end{pmatrix} + \begin{pmatrix}\pm 1\\ 0\end{pmatrix}$$

where $x(0,x_0,y_0) = x_0 \in \mathbb{R}$ and $y(0,x_0,y_0) = y_0 \in \mathbb{R}$.

This leads easily to an explicit representation of the Poincaré mapping $x \mapsto X(x)$ defined as the first return map associated with the horizontal axis. (Though our mapping $x \mapsto X(x)$ is two-valued on the interval $(-1, 1)$, we feel it is more natural than the single-valued return maps associated with the switching lines and the transition or half-return map between the two switching lines (termed as Poincaré mapping by Feng and Loparo [3]).)

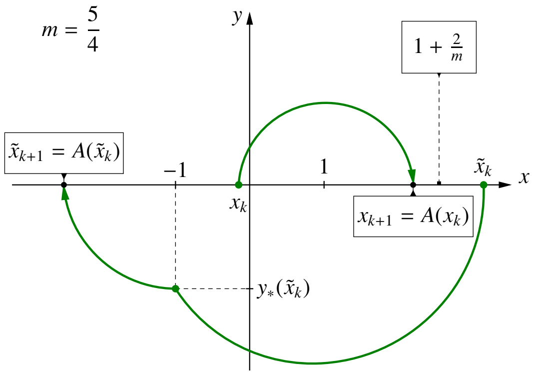

Figure 2: Construction of the Poincaré mapping

For brevity, set $m = e^{\frac{\sigma}{\omega}\pi}$ and, for $x \geq 1 + \frac{2}{m}$, let $y^*(x) = -\sqrt{e^{2\sigma T(x)}(x-1)^2 - 4}$ where $T(x) = T(x, \sigma, \omega) \in (\frac{\pi}{2\omega}, \frac{\pi}{\omega}]$ denotes the (uniquely defined) root of equation

$$e^{\sigma T(x)} \cos(\omega T(x)) = -\frac{2}{x-1} \,. \tag{3}$$

If $x_0 \in (-1, 1 + \frac{2}{m}]$, then the trajectory starting from $(x_0, 0)$ reaches the horizontal axis at

$$X(x_0) = x\left(\frac{\sigma}{\omega}\pi, x_0, 0\right) = -m(x_0 - 1) + 1 \in [-1, +\infty)$$

for the first time. If $x_0 > 1 + \frac{2}{m}$, then the trajectory starting from $(x_0, 0)$ hits the switching line $x = -1$ at the point $(x(T(x_0), x_0, 0), y(T(x_0), x_0, 0)) = (-1, y^*(x_0))$ first and then, after time $T(x_0) + \frac{\sigma}{2\omega}\pi$, it reaches the horizontal axis at

$$X(x_0) = -\sqrt{m} y^*(x_0) - 1 = 2\sqrt{m} \tan(\omega T(x_0)) - 1 \in (-\infty, -1).$$

Finally, we arrive at

$$X(x) = \begin{cases} a(x) & \text{if} \quad x < 1 \\ A(x) & \text{if} \quad -1 < x \end{cases}$$

where

$$A(x) = \begin{cases} -m(x-1) + 1 & \text{if} \quad -1 < x \leq 1 + \frac{2}{m} \\ 2\sqrt{m} \tan(\omega T(x)) - 1 & \text{if} \quad 1 + \frac{2}{m} < x \end{cases}$$

and, by symmetry, $a(x) = -A(-x)$ for all $x < 1$. Note that function $A = A(\cdot, \sigma, \omega) = A(\cdot, \frac{\sigma}{\omega}, 1) : (-1, +\infty) \to \mathbb{R}$ is continuous and strictly decreasing.

Proposition 1 *For $\sigma > 0$, the two-valued mapping $x \mapsto X(x)$ is piecewise expanding.*

Proof. By using the derivative and then the square of equation (3), we obtain that

$$A'(x) = 2\sqrt{m} \cdot \frac{\omega T'(x)}{\cos^2(\omega T(x))}$$

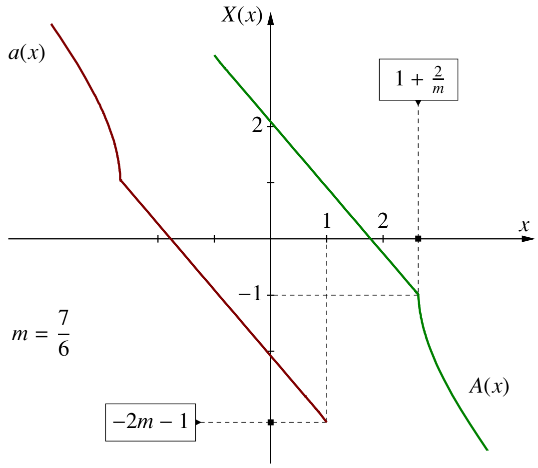

Figure 3: The Poincaré mapping

$$= \sqrt{m} \cdot \frac{4\, e^{-\sigma T(x)}}{\cos^2(\omega T(x)) \cdot (x-1)^2} \cdot \frac{\omega}{\sigma \cos(\omega T(x)) - \omega \sin(\omega T(x))}$$

$$= \sqrt{m} \cdot e^{\sigma T(x)} \cdot \frac{\omega}{\sigma \cos(\omega T(x)) - \omega \sin(\omega T(x))}$$

whenever $x > 1 + \frac{2}{m}$. Is is readily checked that function

$$T \mapsto e^{\sigma T} \frac{\omega}{\sigma \cos(\omega T) - \omega \sin(\omega T)}$$

is negative and strictly decreasing on the interval $(\frac{\pi}{2\omega}, \frac{\pi}{\omega}]$. Thus, $A'(x) \leq \sqrt{m} e^{\sigma \frac{\pi}{2\omega}} \frac{\omega}{0 - \omega 1} = -m < -1$ whenever $x \geq 1 + \frac{2}{m}$. The rest is clear. □

As a simple consequence of the Proposition, equation $A(x) = -x$ has a unique solution $m^* = m^*(\frac{\sigma}{\omega}) > 1 + \frac{2}{m}$ if $m > 1$. The periodic sequence $-m^*, m^*, -m^*, m^*, \ldots$ corresponds to a periodic orbit of system (1)–(2). If also $m^* \geq 2m + 1$, then the restriction of the two-valued mapping $x \mapsto X(x)$ to the interval $[-m^*, m^*]$ is onto. (Note that inequality $m^* \geq 2m + 1$ is satisfied whenever $m = 1 + \varepsilon$, $0 < \varepsilon$ sufficiently small.)

Iterations of the Poincaré mapping $x \mapsto X(x)$ can be started at points where X is single-valued. Clearly $(x_0, x_1) \in \text{Graph}(A)$ whenever $x_0 \geq 1$, and $(x_0, x_1) \in \text{Graph}(a)$ whenever $x_0 \leq -1$. Any time $(x_{k-1}, x_k) \in \text{Graph}(A)$, then $x_{k+1} = A(x_k)$ if $x_k > -1$ and $a(x_k)$ if $x_k \leq -1$. Similarly, any time $(x_{k-1}, x_k) \in \text{Graph}(a)$, then $x_{k+1} = a(x_k)$ if $x_k < 1$ and $A(x_k)$ if $x_k \geq 1$. Thus, iterations of the Poincaré mapping $x \mapsto X(x)$ have to be understood as iterations on $\text{Graph}(a) \cup \text{Graph}(A)$ which, in turn—via the concatenated arclength parametrization of $\text{Graph}(a) \cup \text{Graph}(A)$ below—can be represented as iterations of an associated, standard interval map.

3. The associated interval map

Now we pass to two-valued mappings of the form

$$X(x) = \begin{cases} \varphi(x) & \text{if } x \in [a,b] \\ \Phi(x) & \text{if } x \in [c,d] \end{cases}$$

where constants a, b, c, d satisfy $0 = a < c < b < d$, mappings $\varphi : [a,b] \to [a,d]$ and $\Phi : [c,d] \to [a,d]$ are twice continuously differentiable, strictly increasing, and satisfy inequalities $\varphi(a) < b < \varphi(b)$ and $\Phi(c) < c < \Phi(d)$.

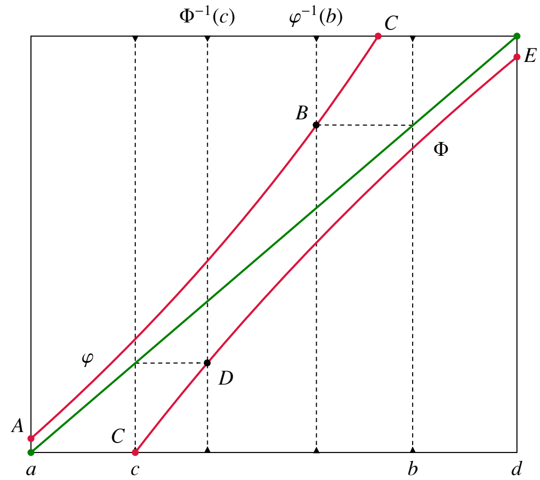

Figure 4: Concatenated arclength parametrization

Concatenated arclength parametrization is introduced via defining

$$\ell(x) = \int_a^x \sqrt{1 + (\varphi'(u))^2}\, du \quad \text{for } x \in [a,b],$$
$$L(x) = C + \int_c^x \sqrt{1 + (\Phi'(u))^2}\, du \quad \text{for } x \in [c,d]$$

where $C = \ell(b)$. Observe that $0 = \ell(a)$, $C = L(c)$ and let

$$A = 0\,,\ B = \ell(\varphi^{-1}(b))\,,\ D = L(\Phi^{-1}(c))\,,\ E = L(d)$$

By the construction, $0 = A < B < C < D < E$. The interval map $S : [A,E] \to \mathbb{R}$ associated with $x \mapsto X(x)$ is defined by letting

$$S(s) = \begin{cases} \ell(\varphi(\ell^{-1}(s))) & \text{if } s \in [A,B] \\ L(\varphi(\ell^{-1}(s))) & \text{if } s \in [B,C] \\ \ell(\Phi(L^{-1}(s))) & \text{if } s \in [C,D] \\ L(\Phi(L^{-1}(s))) & \text{if } s \in [D,E]\,. \end{cases}$$

Clearly $S(s) \in [A,E]$ for all $s \in [A,E]$.

For $x \in [a,c) \cup (b,d]$, the sequence of iterates $\{X^k(x)\}_{k\in\mathbb{N}}$ is defined and satisfies

$$X^k(x) = P(S^k(P^{-1}(x))) \quad \text{for each } k \in \mathbb{N} \qquad (4)$$

where mapping $P : [A,E] \to [a,d]$ is defined by letting

$$P(s) = \begin{cases} \ell^{-1}(s) & \text{if } s \in [A,C] \\ L^{-1}(s) & \text{if } s \in [C,E]\,. \end{cases}$$

(As it is indicated by Fig. 4, P can be visualized as a vertical projection because the parametrization we use identifies interval $[A,E]$ with $\text{Graph}(\varphi) \cup \text{Graph}(\Phi)$. Note that $P^{-1}(x)$ is single-valued for $x \in [a,c) \cup (b,d]$.)

Lemma 1 *Suppose that $\varphi'(x) \geq q > 1$ for each $x \in [a,b]$ and $\Phi'(x) \geq q > 1$ for each $x \in [c,d]$. Then $S'(s) \geq q > 1$ for each $s \in [A,E]$.*

Proof. Consider e.g. the case $s \in [C,D]$. Then $s = L(x)$ for some $x \in (c, \Phi^{-1}(c))$ and $S(s) = \ell(\Phi(x))$. Thus

$$S'(s) = \ell'(\Phi(x)) \cdot \frac{\Phi'(x)}{L'(x)} = \sqrt{1 + (\varphi'(\Phi(x)))^2} \cdot \frac{\Phi'(x)}{\sqrt{1 + (\Phi'(x))^2}}$$

and the desired inequality follows immediately. In fact, the first product term is at least $\sqrt{1+q^2}$ and function $p \mapsto p/\sqrt{1+p^2}$ is increasing on $[q, +\infty)$. □

Theorem 1 *Assume that the conditions of the previous Lemma are satisfied. Then there exists an absolutely continuous probability measure ν on interval $[a,d]$ with the property that*

$$\lim_{n\to+\infty} \frac{1}{n} \#\left\{0 \leq k \leq n-1 \mid X^k(x) \in M\right\} = \nu(M)$$

for ν-almost all $x \in [a,c) \cup (b,d]$ and every Borel set $M \subset [a,d]$.

Proof. In view of the famous Lasota–Yorke Theorem (see e.g. in [2]), the associated interval map $S : [A,E] \to [A,E]$ admits an ergodic, absolutely continuous probability measure μ. Since P and both branches of P^{-1} are Lipschitz, formula

$$\nu(M) = \mu(P^{-1}(M))$$

(whenever M is a Borel subset of interval $[a,d]$)

defines an absolutely continuous probability measure on $[a,d]$.

Birkhoff's Ergodic Theorem gives that

$$\lim_{n\to+\infty} \frac{1}{n} \#\left\{0 \leq k \leq n-1 \mid S^k(s) \in P^{-1}(M)\right\} = \mu(P^{-1}(M)) \qquad (5)$$

for μ-almost all $s \in [A,B) \cup (D,E]$ (actually, for μ-almost all $s \in [A,E]$) and every Borel set $M \subset [a,d]$. Since $x = P^{-1}(s)$ is uniquely defined, the left-hand side of (5) equals to

$$\lim_{n\to\infty} \frac{1}{n} \#\left\{0 \leq k \leq n-1 \mid P(S^k(P^{-1}(x))) \in M\right\}$$
$$= \lim_{n\to\infty} \frac{1}{n} \#\left\{0 \leq k \leq n-1 \mid X^k(x) \in M\right\}$$

whereas the right-hand side of (5) equals to $\nu(M)$. □

If $m = 1 + \varepsilon$ and $0 < \varepsilon$ is sufficiently small, then Theorem 1 holds true for (a suitable restriction of) the Poincaré mapping discussed in the previous section.

Remark 1 *Given a pair of Borel subsets M, N of the set $[a, c) \cup (b, d]$, it is not hard to prove that for each $k \in \mathbb{N}$*

$$P^{-1}(X^{-k}(M)) \cap P^{-1}(N) = S^{-k}(P^{-1}(M) \cap P^{-1}(N)).$$

This implies that mixing properties of measure μ are, in a technical sense, inherited by measure ν.

4. Further results

For $1 < \beta < 2$, the β-hysteresis transformation $X_\beta : [0, 1] \to [0, 1]$ is defined as the two-valued mapping

$$X_\beta(x) = \begin{cases} \beta x & \text{if } 0 \le \beta x \le 1 \\ \beta(x-1)+1 & \text{if } \beta - 1 \le \beta x \le \beta \end{cases}$$

The associated interval map (normed to be defined on the unit interval) $S_\beta : [0, 1] \to [0, 1]$ is given by letting

$$S_\beta(s) = \begin{cases} \beta s & \text{if } 0 \le \beta s \le 1/2 \\ \beta(s-1/2)+1 & \text{if } 1/2 \le \beta s \le \beta/2 \\ \beta(s-1/2) & \text{if } \beta/2 \le \beta s \le \beta - 1/2 \\ \beta(s-1/2)+1 & \text{if } \beta - 1/2 \le \beta s \le \beta\,. \end{cases}$$

The interval map S_β, $1 < \beta < 2$, belongs to the class of mappings investigated in a recent paper by Góra [5]. His matrix $\mathcal{S}$ is a 4×4 matrix in our case and, as it is readily checked by a direct computation, $\frac{1}{\beta}$ is not an eigenvalue of $\mathcal{S}$. Hence the absolutely continuous invariant probability measure μ_β associated with S_β is unique and ergodic. The (non-normalized) density function of μ_β can be given

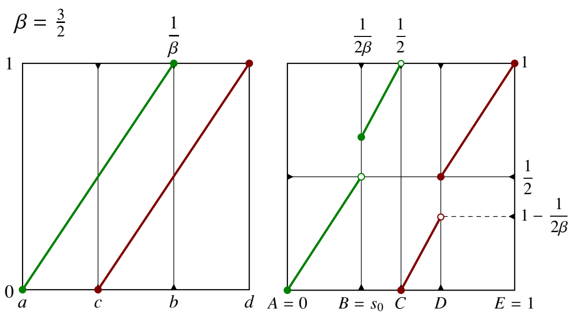

Figure 5: The β-hysteresis transformation and its associated interval map normed to be defined on the unit interval

explicitly as

$$h_\beta(s) = \frac{1}{D\beta} + \frac{1}{\beta^2} + \frac{2}{\beta^2(\beta-1)} + \sum_{k=1}^{\infty} \left(\chi_{[0,1-S^k_{\beta,r}(s_0)]}(s) + \chi_{[S^k_{\beta,r}(s_0),1]}(s) \right) \frac{1}{\beta^{k+1}}.$$

Here D is an appropriate constant (computed from Góra's matrix $\mathcal{S}$: his D_i's are all equal in our case), $\chi_M(s) = 1$ if $s \in M$ and $\chi_M(s) = 0$ if $s \notin M$ (the characteristic function of M), $S_{\beta,r}$ is the right-continuous—in Góra's terminology, the "lazy"—single-valued selection of S_β, and $s_0 = \frac{1}{2\beta}$ (the first point of discontinuity of S_β). For details, see Theorem 8 of Góra [5] as well as our forthcoming paper.

Remark 2 *For each interval $J \subset [0, 1]$ and parameter $1 < \beta < 2$, it is easy to check that $S^k_\beta(J) = [0, 1]$ whenever $k \ge k(J,\beta)$ with some integer $k(J,\beta)$. Hence ([2], p.167) S_β is exact and Bernoulli. The simplest choice of β for which S_β is Markov is $\beta = \frac{1+\sqrt{5}}{2}$, a golden ratio number. This corresponds to the special case $\varphi(b) = c$, $\Phi(c) = b$ in Section 3 and gives rise to beautiful subshift representations.*

Details and complete proofs will be published elsewhere. For combinatorial properties of chaos in piecewise linear maps with hysteresis, we refer to Berkolaiko [1] and to the last Remark of the present note. For 1D chaos in electrical circuits, see the survey paper by Sharkovsky & Chua [7].

Acknowledgement

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