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Combinatorial Configuration Spaces

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Abstract—The notion of cellular stratified spaces was introduced in [BGRT] with the aim of constructing a cellular model of the configuration space of a sphere. In particular, it was shown that the classifying space (order complex) of the face poset of a totally normal regular cellular stratified space X can be embedded in X as a deformation retract.

Here we elaborate on this idea and prove an extension of one of the main results in [BGRT]. We construct an acyclic category, called the face category, $F(X)$ from a totally normal cellular stratified space X. We show the classifying space of $F(X)$ can be embedded into X as a strong deformation retract. As an application, we construct a combinatorial model for the configuration space $\mathrm{Conf}_k(\Gamma)$ of k distinct points for any graph (1-dimensional finite cell complex) Γ.

1. Introduction

Consider the following problem:

Problem 1.1 *Given a space X, construct a combinatorial model for the configuration space $\mathrm{Conf}_k(X)$ of k distinct points in X. In other words, find a cell complex or a simplicial complex $C_k(X)$ that can be embedded in $\mathrm{Conf}_k(X)$ as a Σ_k-equivariant deformation retract.*

Several solutions are known in special cases.

Example 1.2 *For a CW-complex X of dimension 1, i.e. a graph, Abrams constructed a subspace $C_k^{\mathrm{Abrams}}(X)$ contained in $\mathrm{Conf}_k(X)$ in his thesis [Abr00] and proved that*

$$C_k^{\mathrm{Abrams}}(X) \simeq \mathrm{Conf}_k(X)$$

as long as the following two conditions are satisfied:

1. *each path connecting vertices X of valency more than 2 has length at least $k+1$, and*
2. *each homotopically essential path connecting a vertex to itself has length at least $k+1$.*

Here a path means a sequence of composable 1-cells. □

Example 1.3 *Consider the case $X = \mathbb{R}^n$. Define*

$$H_{i,j} = \left\{(x_1,\ldots,x_k) \in \mathbb{R}^k \;\middle|\; x_i = x_j\right\},$$

then it defines an affine subspace $H_{i,j}\otimes\mathbb{R}^n$ in $\mathbb{R}^k\otimes\mathbb{R}^n = (\mathbb{R}^n)^k$ and we have

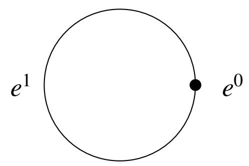
$$\mathrm{Conf}_k(\mathbb{R}^n) = X^k - \bigcup_{1\le i<j\le k} H_{i,j}\otimes\mathbb{R}^n.$$

When $n = 2$, the construction due to Salvetti [Sal87] gives us a regular cell complex $\mathrm{Sal}(\mathcal{A}_{k-1})$ embedded in $\mathrm{Conf}_k(\mathbb{R}^2)$ as a Σ_k-equivariant deformation retract.

More generally, the construction sketched at the end of [BZ92] by Björner and Ziegler and elaborated in [DCS00] by De Concini and Salvetti gives us a regular cell complex $\mathrm{Sal}^{(n)}(\mathcal{A}_{k-1})$ embedded in $\mathrm{Conf}_k(\mathbb{R}^n)$ as a Σ_k-equivariant deformation retract.

This construction is a special case of the construction of a regular cell complex whose homotopy type represents the complement of the subspace arrangement associated with a real hyperplane arrangement. □

There are pros and cons in these two constructions. The conditions for Abrams' model require us to subdivide a given 1-dimensional CW-complex finely. For example, his construction fails to give the right homotopy type of the configuration space $\mathrm{Conf}_2(S^1)$ of two points in S^1 when it is applied to the minimal cell decomposition: $S^1 = e^0 \cup e^1$.



Another problem is that his theorem is restricted to 1-dimensional CW-complexes, although the construction itself works for any cell complex[1].

On the other hand, the crucial deficiency of the second construction is that it works only for $X = \mathbb{R}^n$. An important point suggested by the second construction is that we should work with more general stratifications than cell complexes. The complex $\mathrm{Sal}^{(n)}(\mathcal{A}_{k-1})$ is constructed from the combinatorial structure of the "cell decomposition" of $\mathbb{R}^n\otimes\mathbb{R}^k$ defined by the hyperplanes in the arrangement $\mathcal{A}_{k-1}$ together with the standard framing in $\mathbb{R}^k$. "Cells" in this decomposition are unbounded regions in $\mathbb{R}^n\otimes\mathbb{R}^k$ and the decomposition is not a cell decomposition in the usual sense.

[1] It seems the study of higher dimensional cases has just started [AGH].

One of the motivations of this paper is to find a common framework for working with configuration spaces and complements of arrangements.

2. Cellular Stratified Spaces

Let us first define stratifications and cells.

Definition 2.1 *Let X be a topological space. A* stratification *on X is a a collection C of subsets, called* strata*, satisfying the following properties:*

1. $X = \bigcup_{e \in C} e$.
2. *For $e, e' \in C$, $e \cap e' = \emptyset$ if $e \neq e'$.*
3. *Each stratum e is locally closed, i.e. it is open in $\overline{e}$.*

Definition 2.2 *Let X be a topological space. For a non-negative integer n, an n-*cell structure *on a subspace $e \subset X$ is a pair (D, φ) of a subspace D of the n-disk D^n and a continuous map*

$$\varphi : D \longrightarrow X$$

satisfying the following conditions:

1. $\mathrm{Int}(D^n) \subset D$.
2. *$\varphi(D) = \overline{e}$ and the map $\varphi : D \to \overline{e}$ is a quotient map.*
3. *The restriction $\varphi|_{\mathrm{Int}(D^n)} : \mathrm{Int}(D^n) \to e$ is a homeomorphism.*
4. *The pair (D, φ) is maximal in the poset of pairs satisfying the above conditions for e under inclusions.*

For simplicity, we refer to an n-cell structure (D, φ) on e by e when there is no risk of confusion.

The map φ is called the characteristic map *of e and D is called the* domain *of e. The dimension n of the domain D is called the* dimension *of e.*

Definition 2.3 *Let X be a topological space. A* cellular stratification *on X is a pair (C, Φ) of a stratification $C = \{e_\lambda\}_{\lambda \in \Lambda}$ on X and a collection of cell structures $\Phi = \{\varphi_\lambda : D_\lambda \to \overline{e_\lambda}\}_{\lambda \in \Lambda}$ satisfying the condition that, for each n-cell e_λ, the boundary $\partial e_\lambda = \overline{e_\lambda} - e_\lambda$ is covered by cells of dimension less than or equal to $n - 1$.*

A cellular stratified space *is a triple (X, C, Φ) where (C, Φ) is a cellular stratification on X. As usual, we abbreviate it by (X, C) or X, if there is no danger of confusion.*

In order to be practical, we need to impose certain niceness conditions on cellular stratified spaces. CW-complexes are defined to be cell complexes satisfying the closure finiteness and having the weak topology. Analogously we usually impose the corresponding two conditions on cellular stratified spaces.

Definition 2.4 *A* CW cellular stratified space *is a cellular stratified space X satisfying the following conditions:*

1. *(closure finite) for each n-cell e, its boundary ∂e is covered by a finite number of cells, and*
2. *(weak topology) the topology on X is given by the weak topology determined by the covering given by the closures of all cells.*

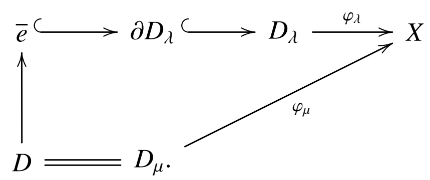
Definition 2.5 *Let X be a cellular stratified space.*

- *We say X is* regular *if all cells in X are regular.*
- *We say X is called* normal *if, for each n-cell e_λ, ∂e_λ is a union of cells of dimension less than or equal to $n - 1$.*

The following definition was introduced in [BGRT] in order to describe a condition under which the order complex of the face poset of a regular cellular stratified space is homotopy equivalent to the original space.

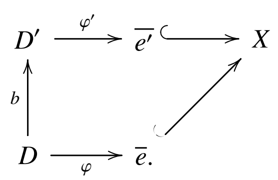
Definition 2.6 *Let (X, C) be a cellular stratified space. X is called* totally normal *if, for each n-cell e_λ,*

- *there exist a structure of regular cell complex on S^{n-1} containing $\partial D_\lambda = D_\lambda - \mathrm{Int}(D^n)$ as a stratified subspace of S^{n-1}, and*
- *for each cell e in ∂D_λ, there exists a cell e_μ in X such that e_μ and e share the same domain and the characteristic map of e_μ factors through D_λ via the characteristic map of e:*



Definition 2.7 *For a cellular stratification C on a space X, define a topological category $F(X, C)$ as follows. Objects are cells $F(X, C)_0 = \{e \mid e \text{ is a cell in } C\}$. $F(X, C)_0$ is equipped with the discrete topology.*

A morphism from a cell $\varphi : D \to \overline{e}$ to another cell $\varphi' : D' \to \overline{e'}$ is a lift of the characteristic map φ of e, i.e. a map $b : D \to D'$ making the following diagram commutative



Note that the existence of a morphism $b : e \to e'$ implies $\overline{e} \subset \overline{e'}$. The set of morphisms $F(X, C)(e, e')$ from e to e' is topologized by the compact open topology as a subspace of $\mathrm{Map}(D, D')$. The composition is given by the composition of maps.

This topological category $F(X, C)$ is called the face category *of C. It is denoted by $F(X)$ or $F(C)$, when C or X is obvious from the context.*

It is straightforward to verify the following.

Lemma 2.8 *$F(X,C)$ is an acyclic category. When (X,C) is totally normal, each $F(X,C)(e,e')$ has the discrete topology. Furthermore when (X,C) is regular, $F(X,C)$ is a poset.*

In general, any acyclic category C has an associated poset $P(C)$ together with a functor $\pi : C \to P(C)$. It is called the underlying poset of C.

Definition 2.9 *For a cellular stratified space (X,C), the underlying poset of the face category $F(X,C)$ is denoted by $P(X,C)$ and is called the* face poset *of (X,C).*

Recall that the order complex of the face poset of a regular cell complex X is the barycentric subdivision of X. The analogue of the order complex construction for topological categories is the classifying space construction.

Definition 2.10 *For a topological category C, let C_0 and C_1 be the spaces of objects and morphisms. The source and the target maps are denote by $s,t : C_1 \to C_0$, respectively. Define*

$$N_n(C) = \{(u_1,\ldots,u_n) \in C_1^n \mid s(u_i) = t(u_{i+1}), 1 \le i \le n-1\}.$$

*An element of $N_n(C)$ is called an n-*chain *of C. The collection $N(C) = \{N_n(C)\}_{n\ge 0}$ together with the face and degeneracy operators defined by the compositions and identity morphisms forms a simplicial space. The geometric realization of $N(C)$ is denoted by BC and is called the* classifying space *of C.*

For each topological space X, there is a standard way to associate a simplicial set. Here we modify the definition and make it into a simplicial space.

Definition 2.11 *For a topological space X, define*

$$S_n(X) = \mathrm{Map}(\Delta^n, X)$$

and topologize it by the compact-open topology. The structure of cosimplicial space on $\{\Delta^n\}_{n\ge 0}$ makes $\{S_n(X)\}_{n\ge 0}$ into a simplicial space. The resulting simplicial space is denoted by $S(X)$ and is called the singular simplicial space *of X.*

The case of regular cell complex suggest the following notation.

Definition 2.12 *Let (X,C) be a totally normal cellular stratified space. Define its* barycentric subdivision $\mathrm{Sd}(X,C)$ *to be the classifying space of the face category*

$$\mathrm{Sd}(X,C) = BF(X,C).$$

Proposition 2.13 *Let C be a CW totally normal cellular stratification on X. Then there exists a map of simplicial spaces*

$$i : N(F(C)) \longrightarrow S(X)$$

such that the composition

$$\tilde{i} : \mathrm{Sd}(X,C) = |N(F(C))| \xrightarrow{|i|} |S(X)| \xrightarrow{\mathrm{ev}} X$$

is an embedding.

The construction of i is based on the map of simplicial sets

$$N(\pi) : N(F(C)) \longrightarrow N(P(C))$$

induced by π. For each n-chain $\boldsymbol{e} = (e_{\lambda_0} < \cdots < e_{\lambda_n})$ of $P(C)$, we have

$$N(\pi)_n^{-1}(\boldsymbol{e}) = F(C)(e_{\lambda_{n-1}}, e_{\lambda_n}) \times \cdots \times F(C)(e_{\lambda_0}, e_{\lambda_1})$$

and we have the following decomposition

$$N_n(F(C)) = \coprod_{\boldsymbol{e}\in N_n(P(C))} F(C)(e_{\lambda_{n-1}}, e_{\lambda_n}) \times \cdots \times F(C)(e_{\lambda_0}, e_{\lambda_1}). \quad (1)$$

This observation allows us to extend the embedding constructed for regular cases in [BGRT] to general cases.

Theorem 2.14 *For a CW totally normal cellular stratification C on X, the above map $\tilde{i} : \mathrm{Sd}(X,C) \to X$ embeds $\mathrm{Sd}(X,C)$ in X as a deformation retract. Furthermore the deformation retraction is natural with respect to morphisms of cellular stratified spaces.*

Again the decomposition (1) allows us to extend the deformation retraction constructed in [BGRT] for regular cases. And we obtain the above theorem.

3. Cellular (Simplicial) Models for Configuration Spaces

Theorem 2.14 gives us a good simplicial model for the configuration spaces of graphs. The starting point is the following observation.

Lemma 3.1 *Any graph Γ, regarded as a 1-dimensional CW-complex, is totally normal. Thus the product cell decomposition on Γ^k is also totally normal.*

Recall the hyperplanes $H_{i,j}$ in $\mathbb{R}^k$ used in Example 1.3. The collection $\mathcal{A}_{k-1} = \{H_{i,j} \mid 1 \le i < j \le k\}$ is called the *braid arrangement* of rank $k-1$. As we have done in [BGRT] for spheres, the braid arrangements can be used to subdivide the product stratification Γ^k in such a way the resulting stratification contains $\mathrm{Conf}_k(\Gamma)$ as a stratified subspace.

Note that a hyperplane H in $\mathbb{R}^n$ cuts $\mathbb{R}^n$ into two parts. In general, a real hyperplane arrangement in $\mathbb{R}^n$ defines a cellular stratification on $\mathbb{R}^n$.

Definition 3.2 *Let Γ be a graph with cellular stratification C. Define a subdivision of the product stratification C^k as follows.*

1. *Let* $\{e^0_\lambda\}_{\lambda\in\Lambda_0}$ *and* $\{e^1_\lambda\}_{\lambda\in\Lambda_1}$ *be the sets of* 0*-cells and* 1*-cells in* Γ*, respectively. Choose linear orders on* Λ_0 *and* Λ_1.

2. *A cell in* Γ^k *is of the form* $e^{\varepsilon_1}_{\lambda_1}\times\cdots\times e^{\varepsilon_k}_{\lambda_k}$ *with* $\varepsilon_1,\cdots,\varepsilon_k = 0$ *or* 1*. Choose a permutation* $\sigma\in\Sigma_k$ *with*

$$\begin{aligned}&(e^{\varepsilon_1}_{\lambda_1}\times\cdots\times e^{\varepsilon_k}_{\lambda_k})\sigma\\ =\;&(\text{a product of 0-cells})\times(e^1_{\mu_1})^{m_1}\times\cdots\times(e^1_{\mu_\ell})^{m_\ell}\end{aligned}$$

   and $\mu_1<\cdots<\mu_\ell$.

3. *Subdivide each* $(e^1_{\mu_j})^{m_j}$ *by the braid arrangement* $\mathcal{A}_{m_j-1}$ *under the identification* $(e^1_{\mu_j})^{m_j}\cong\mathbb{R}^{m_j}$.

The resulting stratification is called the braid stratification *and is denoted by* $\mathcal{B}_k(\Gamma)$.

The braid stratification is designed to include $\mathrm{Conf}_k(\Gamma)$ as a stratified subspace.

Lemma 3.3 *The braid stratification* $\mathcal{B}_k(\Gamma)$ *is invariant under the action of the symmetric group* Σ_k *and contains the configuration space* $\mathrm{Conf}_k(\Gamma)$ *as a* Σ_k*-equivariant stratified subspace.*

Furthermore as a subdivision of a totally normal stratification, $\mathcal{B}_k(\Gamma)$ *and its restriction to* $\mathrm{Conf}_k(\Gamma)$ *are totally normal.*

Definition 3.4 *For a graph* Γ*, define*

$$C^{\mathrm{braid}}_k(\Gamma)=B\left(F\left(\mathcal{B}_k(\Gamma)|_{\mathrm{Conf}_k(\Gamma)}\right)\right).$$

This is our combinatorial model for $\mathrm{Conf}_k(\Gamma)$. Theorem 2.14 guarantees it represents the Σ_k-equivariant homotopy type of $\mathrm{Conf}_k(\Gamma)$.

Corollary 3.5 $C^{\mathrm{braid}}_k(\Gamma)$ *can be embedded in* $\mathrm{Conf}_k(\Gamma)$ *as a* Σ_k*-equivariant strong deformation retract.*

One of the most important features of our model is its efficiency. We do not have to subdivide graphs to obtain a deformation retraction. This fact is very important when we study the homotopical dimension of configuration spaces.

Definition 3.6 *For a topological space X, the* homotopical dimension *of X, denoted by* $\dim^{\simeq} X$ *is defined to be the minimum of dimensions of CW complexes that are homotopy equivalent to X.*

By analyzing the combinatorial structure of the acyclic category $F(\mathcal{B}_k(\Gamma)|_{\mathrm{Conf}_k(\Gamma)})$, we have the following estimate of the dimension of our model.

Theorem 3.7 *For any finite graph* Γ*, we have*

$$\dim C^{\mathrm{braid}}_k(\Gamma)\le\min\{|V(\Gamma)|,k\},$$

where $V(\Gamma)$ *is the set of vertices in* Γ*. In particular, we have*

$$\dim^{\simeq}\mathrm{Conf}_k(\Gamma)\le\min\{|V(\Gamma)|,k\}.$$

Let L be the set of leaves in a graph Γ, then Γ can be deformed into $\Gamma - L$ by an isotopy, and we have $\mathrm{Conf}_k(\Gamma)\simeq_{\Sigma_k}\mathrm{Conf}_k(\Gamma-L)$. By using the minimal cell decomposition of a given graph and then removing leaves, we obtain an alternative proof of the following theorem of Ghrist [Ghr01] as a corollary to Theorem 3.7.

Corollary 3.8 *For any finite graph* Γ*, let* $v(\Gamma)$ *be the number of essential vertices. Namely* $v(\Gamma)$ *is the number of vertices of valency greater than one in a minimal cell decomposition of* Γ*. Then we have*

$$\dim^{\simeq}\mathrm{Conf}_k(\Gamma)\le\min\{v(\Gamma),k\}.$$

References

[Abr00] Aaron Abrams. *Configuration Spaces and Braid Groups of Graphs*. PhD thesis, University of California at Berkeley, 2000, http://www.math.uga.edu/~abrams/research/papers/thesis.ps.

[AGH] Aaron Abrams, David Gay, and Valerie Hower. Discretized configurations and partial partitions, arXiv:1009.2935.

[BGRT] Ibai Basabe, Jesús González, Yuli B. Rudyak, and Dai Tamaki. Higher topological complexity and homotopy dimension of configuration spaces on spheres, arXiv:1009.1851.

[BZ92] Anders Björner and Günter M. Ziegler. Combinatorial stratification of complex arrangements. *J. Amer. Math. Soc.*, 5(1):105–149, 1992.

[DCS00] C. De Concini and M. Salvetti. Cohomology of Coxeter groups and Artin groups. *Math. Res. Lett.*, 7(2-3):213–232, 2000.

[Ghr01] Robert Ghrist. Configuration spaces and braid groups on graphs in robotics. In *Knots, braids, and mapping class groups—papers dedicated to Joan S. Birman (New York, 1998)*, volume 24 of *AMS/IP Stud. Adv. Math.*, pages 29–40. Amer. Math. Soc., Providence, RI, 2001, arXiv:math.GT/9905023.

[Sal87] Mario Salvetti. Topology of the complement of real hyperplanes in $\mathbf{C}^N$. *Invent. Math.*, 88(3):603–618, 1987.