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On the Schroedinger picture in C*-algebraic quantum theory

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Abstract—We discuss state transitions in C*-algebraic quantum theory and reconsider state changes, usually called the Schrödinger picture in quantum theory. We introduce C*-probability structure and transition probability in C*-algebraic quantum theory. By using them, we define category of state transitions. Next, we explain the historical background of this work related to quantum measurement theory.

1. Introduction

We discuss state transitions in C*-algebraic quantum theory. We define the concept of transition probability in C*-algebraic quantum theory and explain the historical background of this work. The motivation for this paper is twofold. One is to reconsider state changes in C*-algebraic quantum theory, usually called the Schrödinger picture in quantum theory. The other is to connect it with the categorical framework. To achieve these purposes, we introduce the concept of C*-probability structure. In the historical background, we mention quantum measurement theory.

C*-algebraic quantum theory is suitable for the description of quantum systems with infinite degrees of freedom including quantum fields. In quantum systems with infinite degrees of freedom, the nontrivial sector structure emerges, which distinguishes the macroscopic aspect of the system. A C*-probability structure describes the probabilistic nature of the system and specifies sectors involved in the family of situations under consideration. Transition probability is introduced in order to describe the transition between C*-probability structures. In the context of quantum measurement theory, the concept of instrument describes the transition between C*-probability structures and has the axiomatic characterization from the statistical point of view. Typical, nontrivial examples of transition probability are given by the measurement of discrete observables. By contrast, the introduction of instrument is motivated by the operationally valid treatment of the measurement of continuous observables. This is the reason why we actively treat quantum measurement theory.

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2. C*-algebraic quantum theory and transition probability

2.1. C*-algebraic quantum theory

Axiom 1 (C*-probability space [1]). *All the statistical aspects of a physical system* **S** *are registered in a C*-probablity space $(\mathcal{X}, \omega)$, where $\mathcal{X}$ is a C*-algebra and ω is a state on $\mathcal{X}$. Observables of* **S** *are described by self-adjoint elements of the C*-algebra $\mathcal{X}$. On the other hand, the state ω on $\mathcal{X}$ statistically correponds to a physical situation (or an experimental setting) of* **S**.

This axiom declares that we describe a quantum system in the language of noncommutative (quantum) probability theory (see [2] for noncommutative probability theory, and [3, 4] for operator algebras).

For every C*-algebra $\mathcal{X}$, $\mathcal{S}_{\mathcal{X}}$ denotes the state space of $\mathcal{X}$. We use the weak* topology for the dual space $\mathcal{X}^*$ of $\mathcal{X}$. In the weak* topology, the neighboorhoods of $\omega \in \mathcal{X}^*$ are indexed by finite sets of elements $X_1, \cdots, X_n$ of $\mathcal{X}$, and $\varepsilon > 0$: $U_\omega(X_1, \cdots, X_n, \varepsilon) = \{\varphi \in \mathcal{X}^* \mid |\varphi(X_j) - \omega(X_j)| < \varepsilon,\ j = 1, \cdots, n\}$. The weak* topology for $\mathcal{S}_{\mathcal{X}}$ is the restriction of that for $\mathcal{X}^*$ to $\mathcal{S}_{\mathcal{X}}$. We adopt the Borel structure of $\mathcal{S}_{\mathcal{X}}$ generated by open sets in the weak* topology. $\mathcal{B}(\mathcal{S}_{\mathcal{X}})$ denotes the Borel sets of $\mathcal{S}_{\mathcal{X}}$.

The second dual $\mathcal{X}^{**} := (\mathcal{X}^*)^*$ of $\mathcal{X}$ is a W^*-algebra, a C*-algebra which is a dual space of a Banach space. The isometric embedding $\hat{\ }$ of $\mathcal{X}$ into $\mathcal{X}^{**}$ is defined by $\langle \hat{X}, \rho \rangle = \rho(X)$ for all $\rho \in \mathcal{X}^*$. The following axiom is usually assumed.

Axiom 2 (Born statistical formula). *When an observable A of $\mathcal{X}$ is precisely measured in a state ω, the probability* $\Pr\{A \in \Delta \| \omega\}$ *that the spectrum of A belonging to Δ emerge is given by* $\Pr\{A \in \Delta \| \omega\} = \langle E^{\hat{A}}(\Delta), \omega \rangle$.

2.2. C*-probability structure

Let $\mathcal{X}$ be a C*-algebra and $(\pi, \mathcal{H})$ a representation of $\mathcal{X}$. $\boldsymbol{B}(\mathcal{H})$ denotes the set of bounded linear operators on $\mathcal{H}$. A linear functional ω on $\mathcal{X}$ is said to be π-normal if there exists a trace-class operator ρ on $\mathcal{H}$ such that $\omega(X) = \mathrm{Tr}[\pi(X)\rho]$ for all $X \in \mathcal{X}$. $V(\pi)$ denotes the set of π-normal linear functionals on $\mathcal{X}$. Let $\mathcal{M}$ be a von Neumann algebra on a Hilbert space $\mathcal{K}$. $\mathcal{Z}(\mathcal{M})$ denotes the center of $\mathcal{M}$. $\mathcal{M}_*$ denotes the set of ultraweakly continuous linear functionals on $\mathcal{M}$.

A linear subspace $\mathcal{V}$ of $\mathcal{X}^*$ is said to be central if there exists a central projection C of $\mathcal{X}^{**}$, i.e., $C \in \mathcal{Z}(\mathcal{X}^{**})$, such that $\mathcal{V} = C\mathcal{X}^*$ [5]. The dual space $\mathcal{V}^*$ of a central subspace $\mathcal{V}(= C\mathcal{X}^*)$ is a W^*-algebra (isomorphic to $C\mathcal{X}^{**}$). A central subspace is said to be σ-finite if its dual is σ-finite.

Example 1 (See [4, Chapter III] for example)**.** (1) *Let $\mathcal{X}$ be a C*-algebra and $(\pi, \mathcal{H})$ a representation of $\mathcal{X}$. There exists a central projection $C(\pi)$ of $\mathcal{X}^{**}$ such that*

$$V(\pi) = C(\pi)\mathcal{X}^* = \{C(\pi)\varphi \mid \varphi \in \mathcal{X}^*\} = \{\varphi \in \mathcal{X}^* \mid C(\pi)\varphi = \varphi\}. \tag{1}$$

(2) *Let $\mathcal{M}$ be a von Neumann algebra on a Hilbert space $\mathcal{H}$. There exists a central projection C of $\mathcal{M}^{**}$ such that $\mathcal{M}_* = C\mathcal{M}^*$. In particular, $\boldsymbol{B}(\mathcal{H})_*$ is a central subspace of $\boldsymbol{B}(\mathcal{H})^*$.*

Definition 1 (C*-probability structure)**.** *$a = (\mathcal{X}_a, \mathcal{V}_a)$ is called a C*-probability structure if it is the pair of a C*-algebra $\mathcal{X}_a$ and a central subspace $\mathcal{V}_a$ of $\mathcal{X}_a^*$.* **C*-PS** *denotes the class of C*-probability structures. For each $a = (\mathcal{X}_a, \mathcal{V}_a) \in$* **C*-PS***, we put $\mathcal{S}_a = \mathcal{S}_{\mathcal{X}_a} \cap \mathcal{V}_a$.*

Here we adopt the next axiom, a sequel to Axiom 1.

Axiom 3. *A quantum system in physical situations (or experimental settings) contained in a fixed category is statistically specified by a C*-probability structure.*

2.3. Definition

We shall define the concept of transition probability by using C*-probability structure.

Definition 2 (Transition probability)**.** *Let $a, b \in$* **C*-PS***. A map $P(\cdot \leftarrow \cdot) : \mathcal{B}(\mathcal{S}_{\mathcal{X}_b}) \times \mathcal{S}_a \to [0, 1]$ is called a transition probability for (a, b) if it satisfies the following two conditions:*
(1) *For every $\omega \in \mathcal{S}_a$, $P(\cdot \leftarrow \omega)$ is a probability measure on $\mathcal{S}_{\mathcal{X}_b}$.*
(2) *For any pair $\omega \in \mathcal{S}_a$ and $\Delta \in \mathcal{B}(\mathcal{S}_{\mathcal{X}_b})$ such that $P(\Delta \leftarrow \omega) \neq 0$, $\omega_{(P,\Delta)} \in \mathcal{S}_b$, where, for any pair $\omega \in \mathcal{S}_a$ and $\Delta \in \mathcal{B}(\mathcal{S}_{\mathcal{X}_b})$ such that $P(\Delta \leftarrow \omega) \neq 0$, we define a state $\omega_{(P,\Delta)}$ on $\mathcal{X}_b$ by*

$$\omega_{(P,\Delta)}(X) = \int_\Delta \rho(X)\, \frac{dP(\rho \leftarrow \omega)}{P(\Delta \leftarrow \omega)}, \qquad X \in \mathcal{X}_b. \tag{2}$$

When $a = b$, a transition probability for (a, b) is also called a transition probability for a for simplicity. For every one element set $\{\varphi\}$, $P(\{\varphi\} \leftarrow \omega)$ is denoted by $P(\varphi \leftarrow \omega)$.

Example 2 (Deterministic transition)**.** (1) *Let $a_{\mathcal{X}} = (\mathcal{X}, \mathcal{X}^*) \in$* **C*-PS** *and α be a *-automorphism of $\mathcal{X}$. A transition probability $P^{(\alpha)}$ for $a_{\mathcal{X}}$ is defined by $P^{(\alpha)}(\Delta \leftarrow \omega) = \delta_{\omega\circ\alpha}(\Delta)$.*
(2) *Let $a, b \in$* **C*-PS***, and $T : \mathcal{V}_a \to \mathcal{V}_b$ a unital positive linear map. A transition probability $P^{(T)}$ for (a, b) is defined by $P^{(T)}(\Delta \leftarrow \omega) = \delta_{T\omega}(\Delta)$ for all $\omega \in \mathcal{S}_a$ and $\Delta \in \mathcal{B}(\mathcal{S}_{\mathcal{X}_b})$.*

2.4. Composition of transition probabilities

Definition 3. *A transition probability P for (a, b) is said to be discrete if, for every $\omega \in \mathcal{S}_a$, $P(\cdot \leftarrow \omega)$ is a discrete probability measure on $\mathcal{S}_{\mathcal{X}_b}$.*

Definition 4 (Composition)**.** *Let $a, b, c \in$* **C*-PS***, and Q and P be transition probabilities for (b, c) and (a, b), respectively. Suppose that P is discrete. The product $Q * P$ of Q and P is defined as follows: for every $\omega \in \mathcal{S}_a$,*

$$(Q * P)(\Gamma \times \Delta|\omega) = \sum_{\rho \in \Delta \cap S_{P,\omega}} Q(\Gamma \leftarrow \rho)\, P(\rho \leftarrow \omega) \tag{3}$$

for all $\Gamma \in \mathcal{B}(\mathcal{S}_{\mathcal{X}_c})$ and $\Delta \in \mathcal{B}(\mathcal{S}_{\mathcal{X}_b})$.

To extend the product into the case where P is not discrete, we use Riesz-Markov-Kakutani theorem stating the one-to-one correspondence between probability measures on a compact Hausdorff space S and states on the set $C(S)$ of continuous functions on S.

Definition 5 (Composition; continued)**.** *Let $a, b, c \in$* **C*-PS***, and Q and P be transition probabilities for (b, c) and (a, b), respectively. Q and P are composable if the following two conditions hold:*
(1) *For every $\omega \in \mathcal{S}_a$ and net $\{P_\alpha\}_{\alpha\in A}$ of discrete transition probabilities for (a, b) convergent to P, the net $\{Q * P_\alpha(\cdot|\omega)\}_{\alpha\in A}$ of states on $C(\mathcal{S}_{\mathcal{X}_c} \times \mathcal{S}_{\mathcal{X}_b})$ weakly converges to a state on $C(\mathcal{S}_{\mathcal{X}_c} \times \mathcal{S}_{\mathcal{X}_b})$.*
(2) *For every $\omega \in \mathcal{S}_a$, the limit of the net $\{Q * P_\alpha(\cdot|\omega)\}_{\alpha\in A}$ is independent of the choice of the net $\{P_\alpha\}_{\alpha\in A}$ of discrete transition probabilities for (a, b) convergent to P. Then the limit is denoted by $Q * P$.*

When Q and P are composable, we define a transition probability $Q \triangleleft P$ for (a, c), called the composition of Q and P, by $(Q \triangleleft P)(\Gamma \leftarrow \omega) = (Q * P)(\Gamma \times \mathcal{S}_{\mathcal{X}_b}|\omega)$ for all $\omega \in \mathcal{S}_a$ and $\Gamma \in \mathcal{B}(\mathcal{S}_{\mathcal{X}_c})$.

We use this composition to define category of state transitions: Objects and arrows are C*-probability structures and transition probabilities, respectively. The latter must satisfy the associative law of the composition of transition probabilities.

Definition 6 (Category of state transitions)**.** *C is a category of state transitions if it has*

Objects *C*-probability structures $a = (\mathcal{X}_a, \mathcal{V}_a)$, and*

Arrows *$b \leftarrow a : f$ with transition probability P_f for (a, b). For every object a, the identity arrow $a \leftarrow a : 1_a$ of a has a transition probability P_{1_a} for a such that $P_{1_a}(\Delta \leftarrow \omega) = \delta_\omega(\Delta)$ for all $\omega \in \mathcal{S}_a$ and $\Delta \in \mathcal{B}(\mathcal{S}_{\mathcal{X}_a})$.*

The composition of arrows involves that of transition probabilities and satisfies the associative law.

3. Historical remarks and instrument

We assume that $\mathcal{H}$ is a separable Hilbert space. We do not distinguish density operators ρ on $\mathcal{H}$ and normal states $\tilde{\rho}$ on $\boldsymbol{B}(\mathcal{H})$ via the isomorphism $\tilde{\cdot} : \boldsymbol{T}(\mathcal{H}) \to \boldsymbol{B}(\mathcal{H})_*$ such that $\tilde{\rho}(X) = \mathrm{Tr}[\rho X]$ for all $X \in \boldsymbol{B}(\mathcal{H})$. We put $a_{\mathcal{H}} = (\boldsymbol{B}(\mathcal{H}), \boldsymbol{T}(\mathcal{H}))$. By using transition probabilities, state transitions by the measurement of discrete observables in the traditional context are given by the following axiom.

Postulate 1. *Let $A = \sum_{a\in\mathbb{R}} aE^A(\{a\})$ be a discrete observable of $\boldsymbol{B}(\mathcal{H})$ to be measured. When a density operator ρ is prepared, the state $\rho_{\{A=a\}}$ after the measurement is uniquely determined for each $a \in Sp(A;\rho) = \{a \in \mathbb{R} \mid \mathrm{Tr}[E^A(\{a\})\rho] > 0\}$, and the transition probability* Pr *for $a_{\mathcal{H}}$ is given by*

$$\Pr(\Delta \leftarrow \rho) = \sum_{a\in Sp(A;\rho)} \mathrm{Tr}[E^A(\{a\})\rho]\delta_{\rho_{\{A=a\}}}(\Delta). \tag{4}$$

In particular, for every $a \in Sp(A;\rho)$,

$$\Pr(\rho_{\{A=a\}} \leftarrow \rho) = \mathrm{Tr}[E^A(\{a\})\rho]. \tag{5}$$

Postulate 2 (von Neumann-Lüders projection postulate)**.** *For each $a \in Sp(A;\rho)$, $\rho_{\{A=a\}}$ in Postulate 1 is given by*

$$\rho_{\{A=a\}} = \frac{E^A(\{a\})\rho E^A(\{a\})}{\mathrm{Tr}[E^A(\{a\})\rho]}. \tag{6}$$

J. von Neumann [6] considered this postulate only for non-degenerate observables, and Lüders [7] generalized it for the degenerate case. Dirac's transition probability [8] motivated the above postulates. Under the above postulates, we have the following lemma.

Lemma 1. *When ρ is a prepared state and values of A not contained in Δ are ignored, the state $\rho_{\{A\in\Delta\}}$ after the measurement of A is given by*

$$\frac{\sum_{a\in\Delta} E^A(\{a\})\rho E^A(\{a\})}{\mathrm{Tr}[\rho E^A(\Delta)]} = \frac{\left(\sum_{a\in\mathbb{R}} E^A(\{a\})\rho E^A(\{a\})\right)\cdot E^A(\Delta)}{\mathrm{Tr}[\rho E^A(\Delta)]}.$$

For nondegenerate discrete observables, von Neumann [6] derived Postulate 2 from

Postulate 3 (Repeatability hypothesis [6, 9])**.** *If an observable A is measured twice in succesion in the object system, then we get the same value each time.*

From Postulates 1 and 3, we have $\mathrm{Tr}[E^A(\{b\})\rho_{\{A=a\}}] = \delta_{ab}$ for all $a \in Sp(A;\rho)$ and $b \in \mathbb{R}$. Under Postulate 1, Postulate 2 implies Postulate 3.

Nakamura and Umegaki [10] pointed out that the map

$$\mathcal{E}_A : \boldsymbol{B}(\mathcal{H}) \ni X \mapsto \sum_{a\in\mathbb{R}} E^A(\{a\})XE^A(\{a\}) \in \{A\}' \tag{7}$$

is nothing but the conditional expectation of $\boldsymbol{B}(\mathcal{H})$ onto the von Neumann algebra $\{A\}' = \{B \in \boldsymbol{B}(\mathcal{H}) \mid AB = BA\}$, and conjectured that the same argument holds for continuous observables. Arveson [11] proved that their conjecture does not hold. Following those investigations, Davies and Lewis [12] introduced the notion of instrument which describes general state changes caused by the measurement in order to formulate measurement theory *not* based on the repeatability hypothesis (Postulate 3).

Let $\mathcal{V}_1$ and $\mathcal{V}_2$ be central subspaces of the dual spaces of C*-algebras $\mathcal{X}_1$ and $\mathcal{X}_2$, respectively. $P(\mathcal{V}_1, \mathcal{V}_2)$ denotes the set of positive linear maps of $\mathcal{V}_1$ into $\mathcal{V}_2$. Also, $\langle\cdot,\cdot\rangle$ denotes the pairing of $\mathcal{V}_1^*$ and $\mathcal{V}_1$.

Definition 7 (instrument)**.** *Let $a, b \in$ **C*-PS** and $(S, \mathcal{F})$ a measurable space. $\mathcal{I}$ is called an instrument for (a, b, S) if it satisfies the following three conditions:*
(1) *$\mathcal{I}$ is a map of $\mathcal{F}$ into $P(\mathcal{V}_a, \mathcal{V}_b)$.*
(2) *$\langle 1, \mathcal{I}(S)\rho\rangle = \langle 1, \rho\rangle$ for all $\rho \in \mathcal{V}_a$.*
(3) *For every $\rho \in \mathcal{V}_a$, $M \in \mathcal{V}_b^*$ and mutually disjoint sequence $\{\Delta_j\}_{j\in\mathbb{N}}$ of $\mathcal{F}$, $\langle M, \mathcal{I}(\cup_j\Delta_j)\rho\rangle = \sum_{j=1}^{\infty}\langle M, \mathcal{I}(\Delta_j)\rho\rangle$.*
An instrument $\mathcal{I}$ for (a, b, S) is said to be completely positive (CP) if $\mathcal{I}(\Delta)$ is completely positive for all $\Delta \in \mathcal{F}$.

Davies and Lewis [12] defined instrument more abstractly. Their definition uses "state space" and includes our definition in some sense. However, we cannot reach our definition from their one without the understanding for sector theory [13, 14]. The theory of CP instrument was developed in [15, 16] in the von Neumann algebraic setting. The theory in the setting of the paper is a future task.

We shall define category of instruments. As defined below, instruments become arrows in the category.

Definition 8 (Category of instruments)**.** *$\mathcal{C}$ is a category of state transitions if it has*

Objects *C*-probability structures $a = (\mathcal{X}_a, \mathcal{V}_a)$, and*

Arrows *$b \leftarrow a : \mathcal{I}$ is an instrument $\mathcal{I}$ for $(a, b, \mathbb{R}^d)$, where $d = 0, 1, 2, \cdots$.*

The composition of arrows is given by the product of instruments (see [12] for the definition of the product).

4. Central instrument

In the C*-algebraic setting, there exists a nontrivial example of instrument, called a central instrument. It gives the simultaneous central decomposition of states belonging to the given central subspace. Thus the unification of sector theory and quantum measurement theory is achieved by the use of central instruments.

Let $a \in$ **C*-PS**, $(S, \mathcal{F})$ be a measurable space, and $C : \mathcal{F} \to \mathcal{Z}(\mathcal{V}_a^*)$ a projection valued measure (PVM). For every $M_1, M_2 \in \mathcal{V}_a^*$ and $\rho \in \mathcal{V}_a$, we define $M_1\rho M_2 \in \mathcal{V}_a$ by $\langle M, M_1\rho M_2\rangle = \langle M_2 M M_1, \rho\rangle$ for all $M \in \mathcal{V}_a^*$. An instrument $\mathcal{I}_C$ for (a, S) is defined by $\mathcal{I}_C(\Delta)\rho = C(\Delta)\rho$ for all $\rho \in \mathcal{V}_a$ and $\Delta \in \mathcal{F}$.

Theorem 2 ([5, Theorem 10]). *$\mathcal{I} = \mathcal{I}_C$ defined above satisfies the following conditions:*
(1) *$\mathcal{I}(S)\rho = \rho$ for all $\rho \in \mathcal{V}_a$.*
(2) *It is repeatable, i.e., it satisfies $\mathcal{I}(\Delta)\mathcal{I}(\Gamma) = \mathcal{I}(\Delta\cap\Gamma)$ for all $\Delta, \Gamma \in \mathcal{F}$.*
(3) *For every $\rho \in \mathcal{S}_a$ and $\Delta \in \mathcal{F}$, $\mathcal{I}(\Delta)\rho$ and $\mathcal{I}(\Delta^c)\rho$ are mutually disjoint.*
(4) *For every $\Delta \in \mathcal{F}$, $\mathcal{I}(\Delta)$ is $\mathcal{V}_a^*$-bimodule map, i.e., for every $\Delta \in \mathcal{F}$, $\rho \in \mathcal{V}_a$ and $M_1, M_2 \in \mathcal{V}_a^*$,*

$$\mathcal{I}(\Delta)(M_1\rho M_2) = M_1(\mathcal{I}(\Delta)\rho)M_2. \tag{8}$$

Conversely, if an instrument $\mathcal{I}$ for (a, S) satisfies the conditions (2) and (4), then there exists a spectral measure $C : \mathcal{F} \to \mathcal{Z}(\mathcal{V}_a^*)$ such that $\mathcal{I} = \mathcal{I}_C$.

An instrument $\mathcal{I}$ for (a, S) is said to be subcentral if it satisfies the conditions (2) and (4) in Theorem 2. An instrument $\mathcal{I}$ for (a, S) is said to be central if it is the maximum in the set of subcentral instruments defined on a, where the maximum is due to the preorder $\prec$ on instruments defined as follows: For instruments $\mathcal{I}_1$, $\mathcal{I}_2$ for (a, S_1) and (a, S_2), respectively, $\mathcal{I}_1 \prec \mathcal{I}_2$ if $\mathcal{I}_1(\mathcal{F}_1) \subset \mathcal{I}_2(\mathcal{F}_2)$ for all $\rho \in \mathcal{S}_a$, where $\mathcal{I}_i(\mathcal{F}_i)$, $i = 1, 2$, is the subset of $\mathcal{P}(\mathcal{V}_a, \mathcal{V}_a)$ defined by $\mathcal{I}_i(\mathcal{F}_i) = \{\mathcal{I}_i(\Delta_i) \mid \Delta_i \in \mathcal{F}_i\}$.

Theorem 3 ([5, Theorem 11]). *$\mathcal{I}_C$ is central if and only if the abelian von Neumann algebra generated by $\{C(\Delta) \mid \Delta \in \mathcal{F}\}$ is isomorphic to $\mathcal{Z}(\mathcal{V}_a^*)$.*

5. Discussion and perspective

The content of the paper can be summarized as the following axiom.

Axiom 4. *A quantum system is specified by a category; its objects are C^*-probability structures and its arrows describe transitions between them. Category of state transitions and that of instruments are such examples.*

The paper [17] by Saigo et al. motivates this work and suggests further development. For example, we do not treat the composite system related to the complete positivity of instrument in the paper yet. The concept of transition probability has room for development. We believe that it is important to establish the formulation of category of state transitions applicable to quantum field theory in the future.

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