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Bifurcation analysis of the transition from the standing pulse wave to the propagating pulse wave in a ring of coupled bistable oscillators

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Abstract—In a ring of coupled bistable oscillators, a standing pulse wave appears when the coupling is weak. This standing pulse wave becomes a propagating pulse wave when the coupling strength exceeds a certain cirtical value. We clarify this mechanism using bifurcational analysis of a certain periodic solution.

1. Introduction

A study of propagating wave phenomenon in coupled oscillator systems is one of the familiar topics of research. The existence of propagating pulse wave in the FitzHugh-Nagumo equation, a model of neuronal system, is well known[1]. In addition, various propagating wave phenomena in several systems such as chaotic pulse [2], propagation of phase-inversion wave [3], etc. were investigated. A basic question concerning these systems is the condition under which propagating wave can emerge. It is known as propagation failure phenomenon that propagating wave fails to propagate below a certain critical coupling strength [1].

In our previous study, we found the propagating pulse wave in an inductor-coupled bistable oscillator system. There exists a standing pulse wave for weak coupling case. However, as the value of coupling strength becomes larger than a certain critical value, a propagating pulse wave appears in some parameter region[4]. In this study, we focus on the formation mechanism of propagating pulse wave. We confirm numerically for 6 coupled oscillator case that the *heteroclinic cycle bifurcation of maps* [1](= HCB) induces the invariant circle corresponding to the propagating pulse wave.

[1]The definition of heteroclinic bifurcation based on the heteroclinic cycle can be seen in [5] in case of flow. In our case it is about Poincare maps. So we make a new word "heteroclinic cycle bifurcation of maps (= HCB)" to mention the bifurcation of Fig.4. Also, we call the situation of Fig.4(b) the "critical heteroclinic cycle of maps (= CHC)". Usually, heteroclinic cycle of maps is structurally stable, but CHC is structurally unstable.

2. Fundamental equation and its dynamics

The system equation of a ring of inductor-coupled bistable oscillators can be written in the following with reference to [4] :

$$\begin{aligned} \dot{x}_i &= y_i \\ \dot{y}_i &= -\varepsilon(1-\beta x_i^2 + x_i^4)y_i \\ &\quad -(1-\alpha)x_i + \alpha(x_{i-1} - 2x_i + x_{i+1}) \\ &, \ i = 1, 2, \cdots, n, \quad x_0 = x_N, x_1 = x_{N+1}, \\ &(\cdot = d/dt) \end{aligned} \tag{1}$$

, where N is the number of oscillators. The x_i denotes the normalized output voltage of the i-th oscillator, y_i denotes its derivative. The parameter ε (> 0) shows the degree of nonlinearity. The parameter α ($0 \leqq \alpha \leqq 1$) is a coupling factor; namely $\alpha = 1$ means maximum coupling, and $\alpha = 0$ means no coupling. The parameter β controls amplitude of oscillation. Each isolated oscillator has two steady-states, namely, no oscillation and periodic oscillation depending on the initial condition.

The analysis of modes based on the averaging method or perturbation method for weakly nonlinear cases was extensively performed in the past [6, 7] . However, the solution for non-weak nonlinear cases does not seem to be analyzed; in fact, it is complicated including the propagating pulse wave solution. In this paper, we analyze the propagating pulse wave solution observed for non-weak nonlinear case. The propagating pulse wave solution consists of several adjacent oscillators oscillating with large amplitude, and the part of large amplitude oscillation in the ring array propagates with a constant speed [4]. Such propagating pulse wave seems to be observed in an arbitrary number of coupled oscillators [2]. Hereafter, we will show the results for the 6 coupled oscillator case. In particular, we will investigate one of the mechanisms of the propagating pulse wave formation.

[2]We confirmed the existence of the propagating pulse wave from the $N = 5$ to $N = 100$ cases via computer simulation.

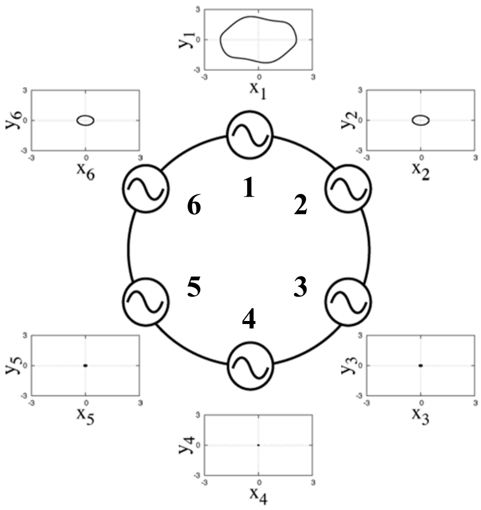


Figure 1: A standing pulse wave (= a periodic solution) occurring in the ring of 6-coupled oscillators. Initial condition: $x_1(0) = 2.0$, $y_1(0) = 0.0$ and $x_k(0) = y_k(0) = 0.0, k = 2, 3, \cdots, 6$. Parameters: $\alpha = 0.05, \beta = 3.2$ and $\varepsilon = 0.36$.

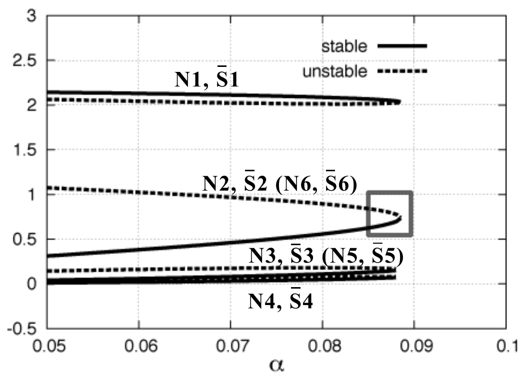
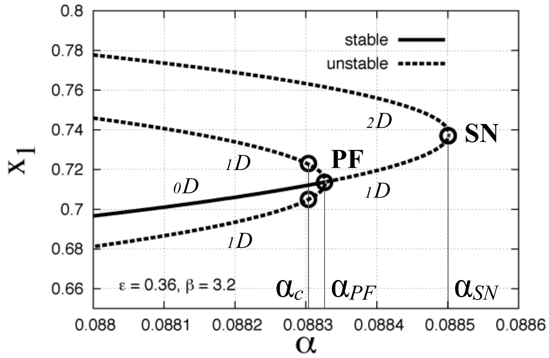
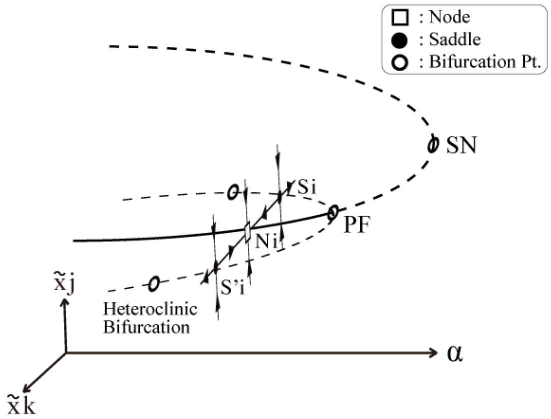


Figure 2: The node Ni with its corresponding saddle $\overline{\mathrm{S}}$i for i = $1, 2, \cdots, 6$ showing the SN bifurcation in terms of α for $\beta = 3.2$ and $\varepsilon = 0.36$. The curve (N2, $\overline{\mathrm{S}}$2) overlaps with the curve (N6, $\overline{\mathrm{S}}$6), and (N3, $\overline{\mathrm{S}}$3) with (N5, $\overline{\mathrm{S}}$5), because they are symmetrically-placed. Subtle structure around the tip region surrounded by a square box is shown in Fig.3(a).



(a)



(b)

Figure 3: Subtle bifurcation diagram around the tip region surrounded by a square in Fig.2 for $\varepsilon = 0.36$ and $\beta = 3.2$. The bifurcation points are as follows: $\alpha_{SN} = 0.088501, \alpha_{PF} = 0.088328$ and $\alpha_c = 0.088302$. The solid curve denotes stable and the dotted curve denotes unstable fixed point. (a)Actual bifurcation diagram. The notation $_mD$ indicates that number of unstable direction of the fixed point is "m". (b) The 3D schematic diagram around the tip region. The axis $\widetilde{x}_j$ denotes the stable direction of saddles and the axis $\widetilde{x}_k$ unstable direction of them. The axes $\widetilde{x}_j$ and $\widetilde{x}_k$ may not correspond to the actual state variables x_j and x_k directly.

3. One of the onset mechanisms of propagating pulse wave

In this system, there exists a certain kind of standing pulse wave solution for small coupling strength α. The standing pulse wave solution is a periodic oscillation, one of which is shown in Fig.1. For simplicity, we fix $\varepsilon = 0.36$ and $\beta = 3.2$ throughout this paper. At first, we investigate this type of standing pulse wave, and then the transition from the standing pulse wave to the propagating pulse wave.

3.1. The standing pulse wave

Figure 1 shows one of the periodic solutions (the standing pulse wave) obtained from the initial condition: $x_1(0) = 2.0$, $y_1(0) = 0.0$ and $x_k(0) = y_k(0) = 0, k = 2, 3, \cdots, 6$ for $\alpha = 0.05$. We choose $\alpha = 0.05$ in order to realize weak coupling so that there exists the standing pulse wave. Since this system has rotational symmetric property (ring coupling structure), other 5 periodic solutions obviously coexist for the same parameters. We define Poincare section as $y_1 = 0$

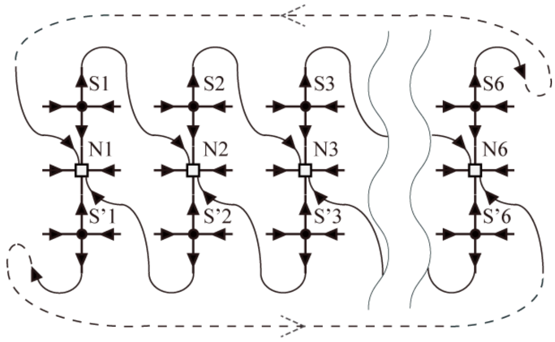
[3], and trace these periodic solutions with respect to α (because the characteristic features of the propagating pulse wave mainly depends on coupling strength [4]). Since the periodic solution becomes a fixed point on the Poincare section, this point becomes a curve when α is varied. In this manner, we can trace 6 periodic solutions as depicted in Fig.2 [8]. The solid curves are mapped points corresponding to the above mentioned stable periodic solutions, namely the nodes (Ni, $i = 1, 2, \ldots, 6$). The dotted curves represent their corresponding saddles ($\overline{\mathrm{S}}\mathrm{i}$, $i = 1, 2, \ldots, 6$)[4].

The node and the corresponding saddle coalesce at a certain value of α. This is called the Saddle-Node (SN) bifurcation point α_{SN}. In this system, there are 6 pairs of (Ni, $\overline{\mathrm{S}}\mathrm{i}$) curves and they disappear simultaneously at the same SN bifurcation point. Namely, the aligned structure of the SN bifurcation points is formed at $\alpha_{SN} \doteqdot 0.088$ in this case. This is rough explanation of bifurcation diagram. In fact, the tip region of each curve presents more sophisticated bifurcations as shown in Fig.3. This is explained in relation to a formation mechanism of the propagating pulse wave in the next section.

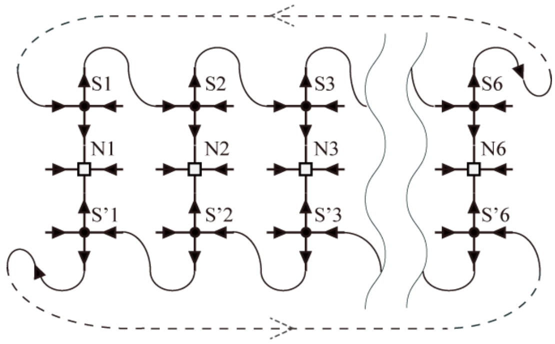
3.2. Propagating pulse wave based on the heteroclinic cycle bifurcation of maps

Figure 3 (a) presents the magnified bifurcation diagram in the square region of Fig.2. From this figure, it is noted that before the SN bifurcation a pitchfork bifurcation (PF) occurs. After the PF bifurcation, a stable node becomes a saddle of index 1 and the corresponding saddle is an index 2 saddle. On the other hand, at the PF point two saddles appear in the backward direction as shown in Fig.3(a); namely, this is the subcritical PF bifurcation. Figure 3 (b) presents the 3D schematic diagram for better understanding the behavior of stable and unstable manifold of saddles in Fig.4. As shown later, there exists a critical $\alpha = \alpha_c$ at which the HCB can occur. This HCB yields a pair of invariant circles (IC), which are the birth of propagating pulse wave going to left and right. Figures 4 (a), (b) and (c) present schematic diagrams representing the connection of Si, Ni and S'i for $i = 1, 2, \cdots, 6$ by unstable manifold (UM)[5] for 3 cases of $\alpha < \alpha_c$, $\alpha = \alpha_c$ and $\alpha > \alpha_c$. For $\alpha < \alpha_c$, UM from S1 goes to N2 and UM from S'1 goes to N6 and vice versa. Therefore, a stable node corresponding to the periodic solution appears. For $\alpha = \alpha_c$, UM from S1 goes to S2 and UM from S'1 goes to S'6 and vice versa, namely a pair of CHCs are formed. For $\alpha > \alpha_c$ the UM emanating from S1 goes to the upper IC and the UM from S'1 goes to the lower IC. These ICs are the *propagating pulse waves just after birth*. Figure 5 demonstrates that the IC just after birth almost follows the CHC. This is a numerical proof of the situation in Fig.4 (c). For other values of β in $3.14 \leqq \beta \leqq 3.25$ we confirmed the similar HCB ensuring generation of IC.

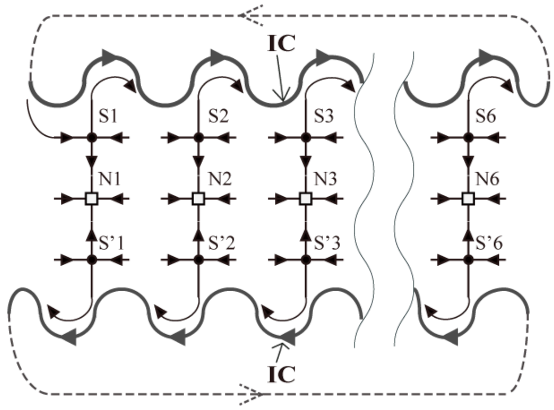
Practically, for $3.14 \leqq \beta \leqq 3.25$, if the initial condition is set on the periodic solution (standing wave solution) and increase α, the periodic solution persists up to $\alpha = \alpha_{PF}$ and for $\alpha > \alpha_{PF}$ the periodic solution jumps to the propagating



(a) $\alpha < \alpha_c$



(b) $\alpha = \alpha_c$



(c) $\alpha > \alpha_c$

Figure 4: Schematic diagram of the relation between the nodes, saddles with their UM for three cases of α. The square box represents each node and its corresponding saddles are drawn as the black circle. The thin curves with arrows represent the UM and the thick ones in (c) denote the invariant circle (IC) corresponding to the propagating pulse wave.

[3] We take mapped points when the flow penetrates the hyper-plane from + to −.

[4] This saddle is index 2, at least, for $\alpha \geqq 0.05$.

[5] The shape of UM is obtained, together with the compensation algorithm, by repeating the mapping of which initial value is chosen on the unstable eigenvector[9]. The method to obtain initial point on the unstable eigenvector is referred to (57) in [10].

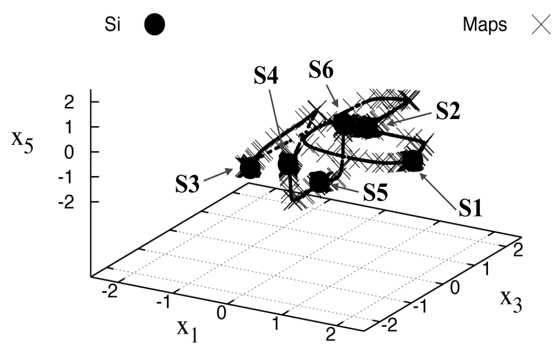


Figure 5: Superimposed presentation of the upper CHC in Fig.4 (b) and IC (mapped points : ×) of the propagating pulse wave just after birth for $\alpha = 0.089$, $\beta = 3.2$ and $\varepsilon = 0.36$. The curve represents the upper CHC connecting 6 saddles for $N = 6$.

wave solution (IC). On the contrary, if the initial condition is set on IC for $\alpha > \alpha_{PF}$ and then decrease α, the IC disappears at $\alpha = \alpha_c$ to become the periodic solution (standing wave solution). Namely, a hysteresis phenomenon between the standing wave solution and the propagating wave solution can be seen in $\alpha_c \leqq \alpha \leqq \alpha_{PF}$.

4. Conclusions

In this paper, we elucidate the onset mechanism of the propagationg pulse wave based on HCB for the ring of 6 bistable oscillators. Namely, there exists the standing pulse wave (the stable periodic solution) for small coupling factor, and beyond a certain coupling factor, it bifurcates to be the propagating pulse wave. It is confirmed for 6 coupled oscillator case that the origin of the propagating pulse wave is the *critical heteroclinic cycle of maps* (= CHC) formed at the parameter near the saddle-node bifurcation point. On the basis of these results, we make conjecture that the CHC connecting N-saddles is one of the general routes of the transition from the standing pulse wave to the propagating pulse wave. In the near future, we will investigate the conjecture of HCB as the onset of the propagating pulse wave for larger number of oscillator cases.

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