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Bifurcation–based synthesis of a PWC analog spiking neuron model

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Abstract—An artificial spiking neuron model which has a *piece-wise constant* (ab. PWC) vector field and state-dependent reset is introduced. Using the analysis techniques of discontinuous ODEs, it is shown that the model can reproduce 4 types of the typical neuron-like responses (neurocomputational properties), the occurrence mechanisms of which have qualitative similarities to those of Izhikevich's simple neuron model.

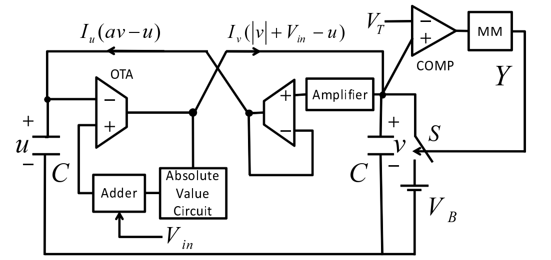
1. Introduction

Neurons exhibit various responses depending on stimulation inputs and parameter values. According to [1, 2], 20 types of typical responses of neurons are called *the most fundamental neuroccomputational properties*. Many mathematical models (e.g., Izhikevich's simple neuronmodel [1]-[3] and Hodgkin-Huxlay's model [4]) have been studied intensively, where Izhikevich's simple neuron model can reproduce all the 20 types of typical responses of neurons. However, since typical control parameters of these mathematical neuron models are nonlinearities of *ordinary differential equations* (ab. ODEs), straightforward analog circuit implementations of these models [5]-[12] are sometimes cumbersome. Hence, as a hardware-oriented neuron model, we have proposed a *piece-wise constant* (ab. PWC) analog spiking neuron model which can be implemented by a simple electronic circuit [13]. The dynamics of the model is described by an ODE with PWC characteristics together with a state-dependent reset. It has been shown that the PWC analog spiking neuron model can reproduce a variety of excitatory responses of neurons [13]. In this paper, It is shown that the model can reproduce typical neuron-like responses (i.e., *phsic spiking*, *phasic bursting*, *class 2 excitable* , *subthreshold oscillation*), where occurrence mechanisms of these responses have qualitative similarities to those of Izhikevich's simple neuron model [1, 2, 3]. Significances of this paper include the following points. **(1)** Advantages of the PWC vector field include: easy to implement by a compact electronic circuit, easy to tune parameter values, and suitability for theoretical analysis based on theories on discontinuous ODEs [14]. **(2)** The neural prosthesis is a recent hot topic, where a typical approach is to prosthesize a damaged part of neural systems by a digital processor [15, 16]. On the other hand, sensory neurons should be prosthesized by analog electronic circuits since sensory neurons accept analog signals and it is not so efficient to utilize digital processor neurons together with analog-to-digital converters to implement them. Due to the advantages in the previous point (1), the model will be a good (compact and tunable) candidate for a sensory neuron prosthesis as well as a hardware pulse-coupled neural network. **(3)** The model can be regarded as a generalized version of a PWC oscillator in [17]-[19]. However, the oscillator is designed as an abstract chaotic oscillator and cannot exhibit neuron-like responses.

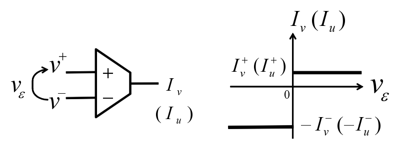
2. Piecewise Constant Analog Spiking Neuron Model

A *piece-wise constant* (ab. PWC) analog spiking neuron model [13] is introduced in Fig.1(a). The model consists of two capacitors, two *operational transconductance amplifiers* (ab. OTAs), a comparator, a monostable multivibrator, an analog switch, an amplifier, an adder, and an absolute value circuit. Fig.1(b) shows the characteristics of the OTA: it outputs a positive (negative) current if the differential voltage $v_\varepsilon = v^+ - v^-$ is positive (negative). From a viewpoint of neuron model, the capacitor voltages v and u can be regarded as an membrane potential and a recovery variable, respectively, as explained in the table in Fig.1. Also, an input voltage V_{in} and a constant voltage V_T can be regarded as a stimulation input and a spiking threshold, respectively. The constant voltage V_T is also regarded as a spike cut-off level [1]. If the membrane potential v reaches the spiking threshold V_T, the comparator (COMP) triggers the monostable multivibrator (MM) to generate a spike $Y = E$. The spike $Y = E$ closes the analog switch S for a short time, and then the membrane potential v is reset to a constant value V_B which is called a reset base. From a viewpoint of neuron model, the spike $Y = E$ is regarded as a firing spike or an action potential as explained in the table in Fig.1. The dynamics of the PWC analog spiking neuron model is described by the following equation.

$$
\begin{cases} C\dot{v} = I_v(|v| + V_{in} - u) \\ C\dot{u} = I_u(av - u) \end{cases} \quad \text{if } v < V_T,
$$
$$
v(t^+) = V_B \quad \text{if } v(t) = V_T,
$$
$$
I_v(v_\varepsilon) = \begin{cases} I_v^+ & \text{if } v_\varepsilon > 0 \\ -I_v^- & \text{if } v_\varepsilon < 0 \end{cases} \tag{1}
$$
$$
I_u(v_\varepsilon) = \begin{cases} I_u^+ & \text{if } v_\varepsilon > 0 \\ -I_u^- & \text{if } v_\varepsilon < 0 \end{cases}
$$
$$
Y(t^+) = \begin{cases} E & \text{if } v(t) = V_T \\ -E & \text{if } v(t) < V_T \end{cases}
$$



(a)



(b)

PWC spiking neuron	Meaning as a neuron model
Capacitor voltage v	Membrane potential
Capacitor voltage u	Recovery variable
Input voltage V_{in}	Stimulation input
Constant voltage V_T	Spiking threshold
Spike-train Y	Output firing spike-train

Figure 1: PWC analog spiking neuron model. (a) Electrical circuit model. COMP and MM represent the comparator and the monostable multivibrator, respectively. (b) Characteristics of the *operational transconductance amplifier* (ab. OTA).

where "$\cdot$" represents the time derivative, t^+ represents $\lim_{\varepsilon\to+0}(t+\varepsilon)$, $I_v^+, I_v^-, I_u^+, I_u^- > 0$ are assumed, and $v(0) \le V_T$ is assumed. In the whole state space

$$\boldsymbol{S} \equiv \{(v,u) | v \le V_T\},$$

the following two borders are defined by the control voltages of the two OTAs (see also Fig.2):

$$\begin{array}{ll} v\text{-nullcline}: & \boldsymbol{\Sigma}_v \equiv \{(v,u) | u = |v| + V_{in}\}, \\ u\text{-nullcline}: & \boldsymbol{\Sigma}_u \equiv \{(v,u) | u = av\}, \end{array}$$

where " $\equiv$ " represents the "definition" hereafter. Since the borders play the same roles as nullclines of a smooth nonlinear ODE, the borders are called v-nullcline and u-nullcline. The nullclines divide the whole state space $\boldsymbol{S}$ into at most four subspaces having the following four vector fields:

$$(\dot{v}, \dot{u}) = \begin{cases} \boldsymbol{V}^{++} \equiv (I_v^+/C, I_v^+/C) & \text{if } u < |v| + V_{in} \text{ and } u < av, \\ \boldsymbol{V}^{-+} \equiv (I_v^-/C, I_v^+/C) & \text{if } u > |v| + V_{in} \text{ and } u < av, \\ \boldsymbol{V}^{+-} \equiv (I_v^+/C, I_v^-/C) & \text{if } u < |v| + V_{in} \text{ and } u > av, \\ \boldsymbol{V}^{--} \equiv (I_v^-/C, I_v^-/C) & \text{if } u > |v| + V_{in} \text{ and } u > av. \end{cases}$$

According to [13, 14], the dynamics of the state (v,u) on the nullclines $\boldsymbol{\Sigma}_v$ and $\boldsymbol{\Sigma}_u$ can be categorized into *sliding mode* and *non-sliding mode* (we also say "*without sliding mode*"). If the mode is categorized into the sliding one, there exists some *sliding vector fields* on the nullclines $\boldsymbol{\Sigma}_v$ and $\boldsymbol{\Sigma}_u$. More detailed explanations of the sliding mode dynamics of the PWC analog spiking neuron model [13] is omitted in this paper due to the page length limitation.

3. Analysis of Typical Neuron-like Responses

In this section, we study 4 types of neuron-like responses of the PWC analog spiking neuron model. Fig.2 shows time waveforms and phase planes of the PWC analog spiking neuron model. In Fig.2, $v + Y'$ is used to show neuron-like waveforms, i.e., spiking wave forms of $v + Y'$ are regarded as action potentials, where

$$Y'(t^+) = \begin{cases} K & \text{if } v(t) = V_T, \\ 0 & \text{if } v(t) < V_T, \end{cases} \qquad (2)$$

and K is a parameter. Fig.3 shows time waveforms and phase planes of Izhikevich's simple neuron model described by the following equation.

$$\begin{cases} \dot{v} = 0.04v^2 + 5v + 140 - u + I \\ \dot{u} = a(bv - u) \end{cases} \qquad (3)$$

$$\text{if } v \ge 30\text{mV}, \text{ then } \begin{cases} v \leftarrow c \\ u \leftarrow u + d \end{cases}$$

We make comparisons between our PWC analog spiking neuron model and Izhikevich's simple neuron model as the followings, where the parameters (C, V_T, K) of the PWC analog spiking neuron model are fixed to $(C, V_T, K) = (0.01, 1.0, 5.0)$.

A. Phasic Spiking : In Fig.2(a), there exists a resting state at first. Next the stimulation input V_{in} is increased, and then the model generates a single spike. Finally, a resting state appeares again. This type of response is called the *phasic spiking* [1, 2]. The above occurrence mechanism of the phasic spiking is qualitatively similar to that of Izhikevich's simple neuron model shown in Fig.3(a).

B. Phasic bursting : In Fig.2(b), there exists a resting state at first. Next the stimulation input V_{in} is increased, and then the model generates bursting spikes. Finally, a resting state appeares again. This type of response is called the *phasic bursting* [1, 2]. The above occurrence mechanism of the phasic bursting is qualitatively similar to that of Izhikevich's simple neuron model shown in Fig.3(b).

C. Class 2 excitable : In Fig.2(c), there exists a resting state at first. Next an stimulation input V_{in} is increased gradually, and the equilibrium point of the resting state loses by *border-collision bifurcation* [14] at the stimulation input $V_{in} = 0$ (we call this border-collision as *saddle-node off invariant circle type border-collision bifurcation* [13] because the bifurcation has qualitative similarities to *saddle-node off invariant circle bifurcation* [20]). Then

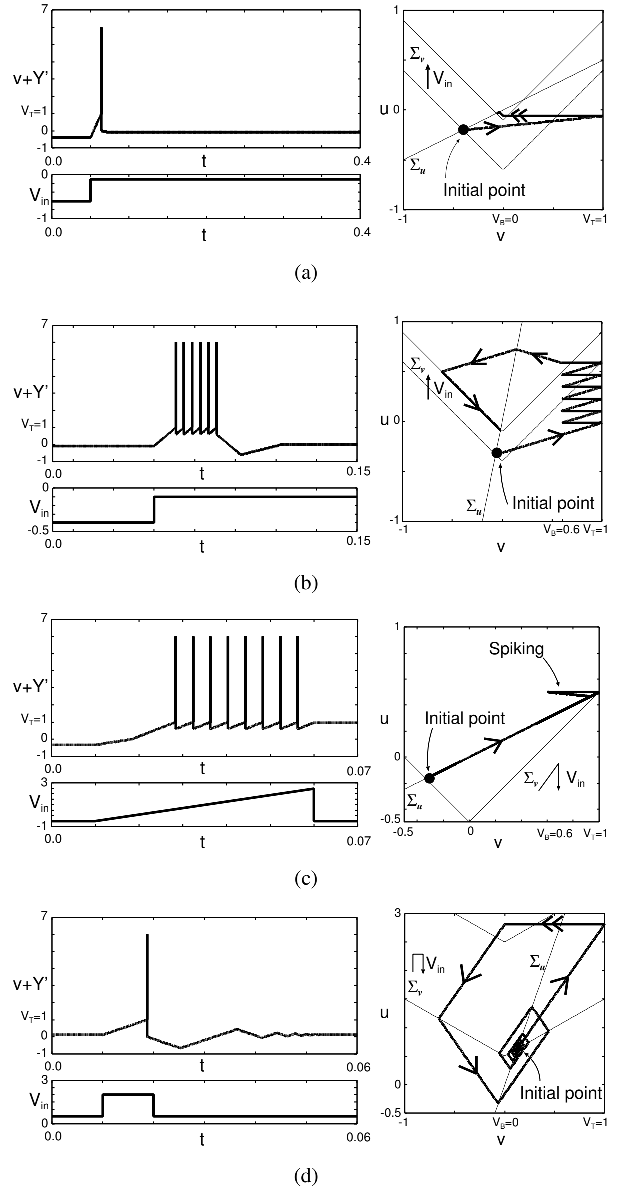


Figure 2: Neuron-like responses of PWC analog spiking neuron model. (a) Phasic spiking. The parameters are $a = 0.5$, $I_v^+ = 1.0$, $I_v^- = 0.1$, $I_u^+ = 0.1$, $I_u^- = 0.1$, $V_B = 0.0$. (b) Phasic bursting. The parameters are $a = 5.0$, $I_v^+ = 1.0$, $I_v^- = 1.0$, $I_u^+ = 0.3$, $I_u^- = 0.3$, $V_B = 0.6$. (c) Class 2 excitable. The parameters are $a = 0.5$, $I_v^+ = 1.0$, $I_v^- = 0.01$, $I_u^+ = 1.0$, $I_u^- = 0.1$, $V_B = 0.6$. (d) Subthreshold oscillation. The parameters are $a = -0.5$, $I_v^+ = 1.0$, $I_v^- = 1.0$, $I_u^+ = 2.5$, $I_u^- = 2.5$, $V_B = -0.0$.

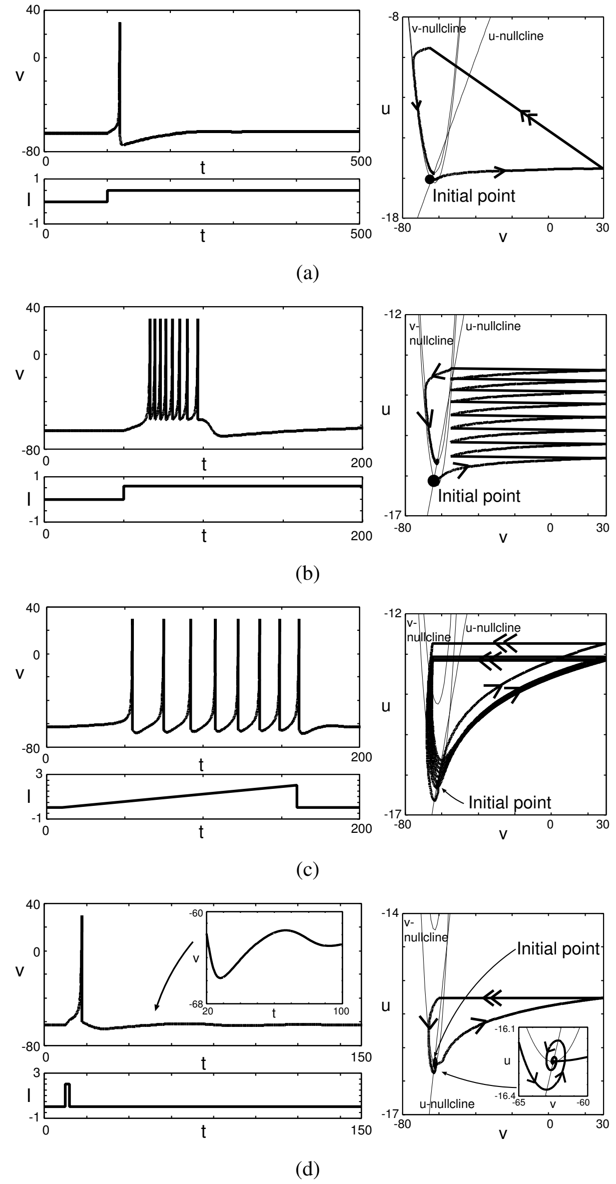


Figure 3: Neuron-like responses of Izhikevich's simple model. (a) Phasic spiking. The parameters are $a = 0.02$, $b = 0.25$, $c = -65$, $d = 6$. (b) Phasic bursting. The parameters are $a = 0.02$, $b = 0.25$, $c = -55$, $d = 0.05$. (c) Class 2 excitable. The parameters are $a = 0.2$, $b = 0.26$, $c = -65$, $d = 0$. (d) Subthreshold oscillation. The parameters are $a = 0.05$, $b = 0.26$, $c = -60$, $d = 0$.

the PWC analog spiking neuron model suddenly begines to generate spikes with a high spike frequency. This type of response is called the *class 2 excitable* [1, 2]. The above occurrence mechanism of class 2 excitable is qualitatively similar to that of Izhikevich's simple neuron model shown in Fig.3(c).

D. Subthreshold oscillation : In Fig.2(d), there exists a resting state at first. Next an excitatory pulse input V_{in} is injected and the PWC analog spiking neuron model generates a single spike. After generating a spike, the PWC analog spiking neuron model shows the oscillation in the subthreshold region. This type of response is called the *subthreshold oscillation* [1, 2]. The above occurrence mechanism of the subthreshold oscillation is qualitatively similar to that of Izhikevich's simple neuron model shown in Fig.3(d).

4. Conclusions

We have introduced the *piece-wise constant* (ab. PWC) analog spiking neuron model. It has been shown that, the model can reproduce the typical neuron-like responses (i.e., *phasic spiking*, *phasic bursting*, *class 2 excitable*, *subthreshold oscillation*), where the occurrence mechanisms of these neuron-like responses have qualitative similarities to those of Izhikevich's simple neuron model. Future problems include: (a) more in-depth theoretical analysis of responses of the model and (b) synthesis of a network of the PWC analog spiking neuron model and investigation of its applications.

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