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# Dynamics of a partially inactivated population of coupled oscillators

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**Abstract**—A large population of coupled nonlinear oscillators may deteriorate due to a variety of causes such as aging and accidents. As a result, it may happen that some elements turn inactive, losing spontaneous oscillatory activity. Here discussed is the behavior of such a population of coupled oscillators when the ratio of inactive elements as well as the coupling strength are varied, with an emphasis on a sort of phase transition, called *aging transition*.

## 1. Introduction

Large populations of coupled nonlinear oscillators have been playing crucial roles in a variety of disciplines of science and technology. Their coherent behaviors such as synchronization have not only given a great impact to the study of diverse rhythmic phenomena, but also suggested a rich variety of potential applications. In view of these, quite a few studies have been done so far on the dynamics of such large-scale dynamical systems[1, 2].

However, there is one point overlooked in such studies, which is the fact that real coupled oscillators, like any other systems, suffer from some kind of deterioration from the beginning or as time passes. Motivated by this, we have theoretically studied the effect of "bad components" on the behavior of a population of coupled oscillators[3, 4], where a "bad component" means an oscillator which has lost the ability of performing self-sustained oscillation. Such a component of the population will be hereafter called an "inactive" oscillator, while a component keeping that ability an "active" one. More specifically, we have examined what happens in such a system as the raio of inactive elements as well as the coupling strength are varied. This problem is important, for example, in understanding the robustness of ubiquitous biological rhythms and in technological contexts, where no system is allowed to be fragile to defects.

The purpose of this article is to briefly review our recent results on this problem. In Section 2, we deal with globally coupled oscillators to encounter such phenomena as aging transition and desynchronization or clustering. Then, we proceed to Section 3, where a case of locally coupled oscillators is examined to check what new features the locality of coupling brings to the system. This paper comes to an end after a summary is given along with some discussion in Section 4.

## 2. The case of global coupling[3, 4]

Here we focus on a large ensemble of globally coupled oscillators, which is assumed to consist of two subpopulations: active oscillators and inactive ones. Let the size ratio of the latter be $p$, which is one of the key parameters. For simplicity, all elements of each subpopulation are set to be identical. Moreover, the global coupling is supposed to be diffusive. The general form of such systems would then be as follows:

$$\dot{\mathbf{x}}_j = \mathbf{F}_j(\mathbf{x}_j) + \frac{K}{N}\sum_{k=1}^{N} D\cdot(\mathbf{x}_k - \mathbf{x}_j) \tag{1}$$

for $j = 1,\ldots,N(\gg 1)$, where the overdot means differentiation with respect to time $t$, $\mathbf{x}_j$ is the state vector of the $j$th element, the first term on the right hand side represents uncoupled dynamics, chosen as $\mathbf{F}_j = \mathbf{F}$ for $j \in \mathrm{S_a} \equiv \{1,\ldots,N(1-p)\}$, $= \mathbf{G}$ for $j \in \mathrm{S_i} \equiv \{N(1-p)+1,\ldots,N\}$, in which $\mathrm{S_a}$ and $\mathrm{S_i}$ are the active and inactive subpopulations, respectively; $K$ is the coupling strength, and $D$ is a diffusion matrix. The uncoupled dynamics of each inactive element, i.e. $\dot{\mathbf{x}} = \mathbf{G}(\mathbf{x})$, is simple: it is stationary with a stable fixed point. In contrast, active dynamics may vary from one system to another. This paper is concerned mostly with a periodic case. Our numerical and analytical results indicate that this type of dynamical systems exhibits a kind of phase transition from a dynamic state to a stationary one, as $p$ exceeds a critical value, say $p_c$ ($< 1$), provided $K$ is greater than a threshold value denoted by $K_c$ hereafter. To illustrate such a transition, which we call an *aging transition*(AT), let us take coupled Stuart-Landau (SL) oscillators as an example:

$$\dot{z}_j = (\alpha_j + i\Omega - |z_j|^2)z_j + \frac{K}{N}\sum_{k=1}^{N}(z_k - z_j) \tag{2}$$

for $j = 1,\ldots,N$, where $z_j$ is the complex amplitude of the $j$th oscillator, $\alpha_j$ is a parameter specifying the distance from a Hopf bifurcation, taken as $\alpha_j = a > 0$ ($j \in \mathrm{S_a}$), $= -b < 0$ ($j \in \mathrm{S_i}$), and $\Omega$ is the natural frequency. For $K = 0$, each active element is a limit-cycle oscillator with amplitude $\sqrt{a}$ and period $2\pi/\Omega$, whereas each inactive one settles at the trivial fixed point, $z = 0$[5]. Numerical results indicate that for $K > 0$, all active elements perfectly synchronize and so do all inactive ones. We therefore reduce

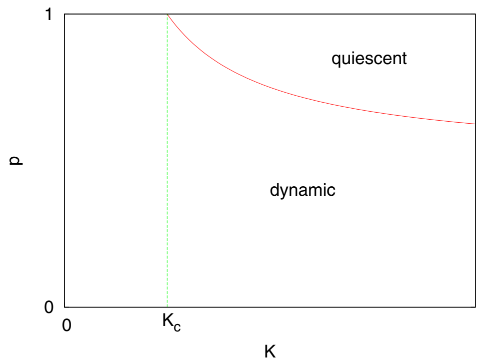


Figure 1: Schematic $(K, p)$ phase diagram. In the dynamic phase, all elements oscillate, while in the quiescent phase, every element is stationary. The boundary between them, given by $p = p_c$, is where the aging transition takes place. Note that such a transition occurs only for $K > K_c$.

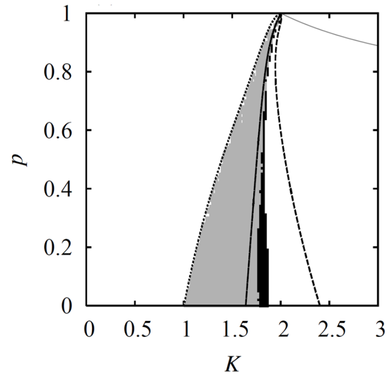


Figure 2: An example of the desynchronization horn in coupled SL oscillators with $a = 2, b = 1, c_2 = -3$. The colored regions are where the active group of the system with $N = 1000$ was numerically found to desynchronize; the state is either periodic (dark) or nonperiodic (gray). The thin dotted curve lying at the right upper corner of each panel is the aging transition line. The dashed, solid, and dotted curves converging at $(K, p) = (2, 1)$ show theoretical curves corresponding to bifurcations of desynchronized states. From Ref.[4].

Eq. (2) to a two-variable system as follows:

$$\dot{A} = (a - Kp + i\Omega - |A|^2)A + KpI \ , \qquad (3)$$
$$\dot{I} = (-b - Kq + i\Omega - |I|^2)I + KqA \ , \qquad (4)$$

where $A$ ($I$) is the common complex amplitude in the synchronized active (inactive) subpopulation. The AT takes place when the trivial fixed point $A = I = 0$ starts to be stable for increasing $p$. A simple algebra yields the following formula of $p_c$ (see Fig. 1):

$$p_c = \frac{a(K + b)}{(a + b)K}. \qquad (5)$$

Since $p_c$ may not be larger than unity, this formula gives $K_c = a$. These results were verified by means of numerical integration of Eq. (3). It is crucial to note the significance of $p_c$: its small values mean that the system's dynamic activity is fragile against the deterioration of elements, so that $p_c$ is a measure of robustness of the system's operation. It is also noteworthy that $p_c$ tends to decrease with $K$. This leads to the warning that increasing the coupling strength to assure the system's coherence may damage its robustness against aging, accidents, and so on. Our studies so far suggest that this warning applies to many coupled-oscillator systems other than SL as well. There is one more point with AT: Suitably defined order parameters to measure the degree of order of the system can be shown to display universal scaling laws near $p = p_c$ and also in the vicinity of the special point $(K, p) = (K_c, 1)$.

The SL oscillator treated above has one nongeneric property as a model of periodic oscillators, which is that it is isochronous, meaning that the frequency $\dot{\theta}$, where $\theta \equiv \mathrm{Arg}(z)$, does not depend on the amplitude $|z|$. This property is a consequence of the fact that the coefficient of the nonlinear term is real in Eq. (2). Nonisochronicity follows by adding an imaginary part, say $ic_2$, to the coefficient. It was found that if $|c_2|$ exceeds the value of one, then there appears a new horn-like region in the $(K, p)$ plane such that the synchronization of active oscillators breaks down, though the inactive subpopulation remains synchronized there. Figure 2 shows an example of this case. We call such a region a *desynchronization horn* (DH). In a DH, active oscillators split into a number of clusters, where a cluster stands for a set of oscillators with identical state vectors. The number of clusters as well as their dynamics depend on not only parameters, but initial conditions. A couple of examples of this phenomenon, clustering, of the active group is displayed in Fig. 3. The existence of a DH in the $(K, p)$ phase diagram also seems to be a faily common feature of oscillator populations belonging to the category of Eq. (1). A heuristic theory was proposed to explain the occurrence of clustering in coupled nonisochronous SL oscillators[6, 4]. Quite recently, effects of nonscalar diffusive coupling and frequency distribution started to be investigated[7].

## 3. The case of local coupling[8]

The global coupling dealt with in the previous section is of course an idealistic limit of long-range coupling. In order to make the theory of "aging" more practical, one needs to elucidate what differences follow if this idealization is loosened. Here, as a first step towards answering this question, we consider the behavior of a partially inactivated population with coupling in the opposite extreme. More specificaly, we suppose that a large number of active

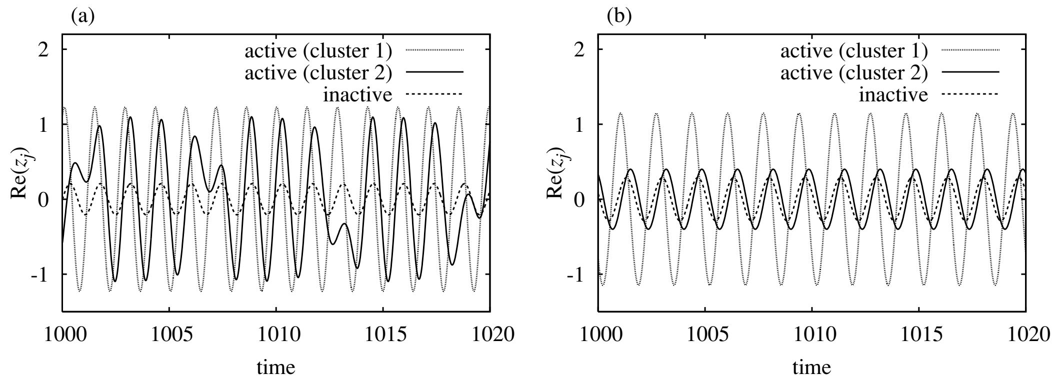


Figure 3: Examples of time series data for $N = 1000, c_2 = -3, a = 2, b = 1, p = 0.4$. Here we have three synchronized clusters: the larger active (cluster 1), the smaller active (cluster 2), and the inactive. (a) $K = 1.32$ (quasiperiodic state; the size of cluster 2 is 2). (b) $K = 1.82$ (periodic state; the size of cluster 2 is 27). From Ref.[4].

or inactive oscillators are placed on a ring with nearest-neighbor interactions. By again taking SL oscillators as an example, we have the following equations:

$$\dot{z_j} = (\alpha_j + i\Omega)z_j - \tilde{c_2}|z_j|^2 z_j + K\tilde{c_1}(z_{j+1} - 2z_j + z_{j-1}) \quad (6)$$

for $j = 1, \ldots, N$ ($\gg 1$) with $\tilde{c_k} \equiv 1 + ic_k$ ($k = 1, 2$) and $z_0 = z_N$, $z_{N+1} = z_1$, where $c_1$ in the interaction term parametrizes the degree of non-scalar nature of the diffusive coupling ($c_1 = 0$ corresponds to scalar coupling). The choices of $\alpha_j$ are the same as before. However, there is an important difference from the case of global coupling: The system's architecture cannot be determined by the aging parameter $p$ alone, so that it is necessary to perform averaging over as many different realizations of $\{\alpha_j\}$ as possible. The results to follow were all obtained after this procedure.

Figure 4 presents simulation results for the aging transition boundary in the $(K, p)$ phase diagram, which results reveal that such a transition exists in the case of local coupling as well, again with a critical coupling strength $K_c$. However, the same figure also indicates that the AT boundaries monotonically shift upwards as the system size $N$ grows, allowing us to conjecture that $p_c$ converges to unity in the limit $N \to \infty$, irrespective of $K$. The next figure, Fig. 5, supports this conjecture, where we clearly see that the critical value of $p$ indeed approaches one obeying a $K$-dependent power law, i.e.

$$1 - p_c(K, N) \propto N^{-\gamma(K)}. \quad (7)$$

Numerical simulation shows that the exponent $\gamma(K)$ tends to decrease towards zero for increasing $K$.

The above power law of $p_c$ remains to be theoretically explained as yet, but the disappearance of the AT in the thermodynamic limit can be understood as follows: As long as $p < 1$, there is a finite probability for an arbitrary large segment containing only active elements to exist in the ring and such a segment must destabilize the quiescent state with $z_j = 0$ for all $j$, because influences from inactive elements which work only at both ends of the segment

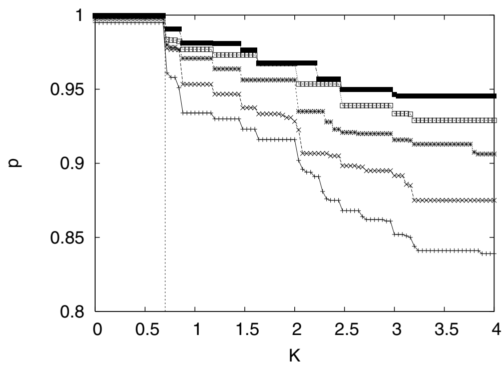


Figure 4: Aging transition boundaries for $a = 2, b = 1, c_1 = -2, c_2 = -1$, where the data are for $N$=100(10), 200(6), 400(6), 800(3), 1600(3) from the lowest to the highest; each parenthesized number is the number of realizations over which averaging was made. The data points are connected by lines to guide the eye. The vertical dotted line shows the theoretical value of $K_c$. From Ref.[8].

should be negligible. Note that this argument applies to higher dimensions as long as the coupling is local. We add that the critical coupling strength $K_c$ can be theoretically obtained (see Fig. 4).

Let us now check how the deterioration or aging of the population laid down on the ring affects its spatiotemporal dynamics. Here we restrict ourselves to the case that stable travelling waves propagate for $p = 0$; this condition may be expressed as $1 + c_1c_2 > 0$ in the continuum limit[5]. Figure 6 is devoted to an example of spatiotemporal phase patterns, in which black regions show where $0 < \mathrm{Arg}(z_j(t)) < \pi$ (mod. $2\pi$). Reflecting the randomness of $\alpha_j$, this kind of patterns for $p > 0$ are more or less irregular, but there is one remarkable fact with them: In spite that the quenched disorder of the system measured by the variance of $\{\alpha_j\}$ is enhanced by increasing $p$ in the range

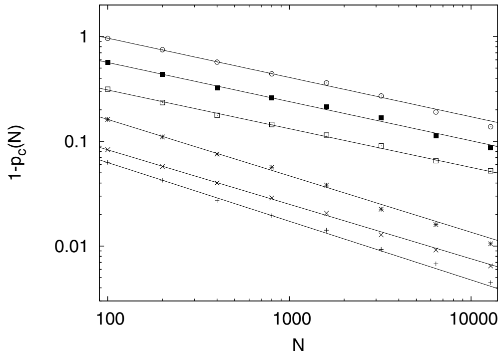


Figure 5: Power law decay of $1 - p_c$ for increasing $N$, where parameter values are the same as in Fig. 4 and $N$=100(100), 200(100), 400(100), 800(100), 1600(100), 3200(50), 6400(50), 12800(20); $K$=1, 1.8, 2.2, 2.6, 3.3, 4 from the lowest data to the highest. The data for $K > 2$ are shifted upwards for clarity. From Ref.[8].

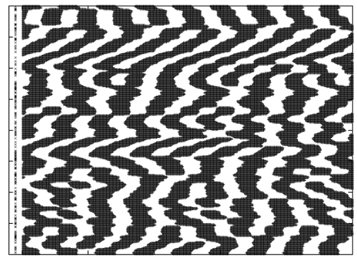

Figure 6: Spatiotemporal behavior of the oscillator phase for $N = 400, K = 2, a = b = 1, c_1 = 1, c_2 = -0.5, p = 0.4$, where the abscissa is time $t$ spanning an interval of size 50, while the ordinate is the oscillator number $j$. The leftmost symbols (×) show the locations of inactive sites. From Ref.[8].

$p < 1/2$, a certain interval of $p$ exists such that the increase in $p$ seems to promote local synchronization of oscillators. This peculiar phenomenon, a kind of *disorder-induced coherence*, can be studied quantitatively through computation of a spatial phase correlation function, the result of which demonstrated that the correlation length becomes minimum at a value of $p$ less than 1/2, in harmony with the behavior of the phase patterns. An intuitive explanation of this phenomenon is given in Ref.[8].

## 4. Summary and discussion

Studies on the dynamics of a large population of coupled active and inactive oscillators have been reviewed. They are important not only in searching for novel phenomena in the new type of dynamical systems, but in the studies of the robustness of natural as well as artificial coupled-oscillator systems against defects. In the case of global coupling, there is a transition from the dynamic phase to the stationary, called an aging transition. In contrast, in the case of local coupling, such a transition is likely to be absent in the thermodynamic limit. Does this imply that AT is not of significance in locally coupled systems ? The answer is definitely NO. As demonstrated above, AT takes place in finite-size systems anyway. Moreover, the size of a real population of coupled oscillators, whether physical or biological, is usually not huge, unlike equilibrium statistical-mechanical systems such as magnets. For example, the entity of mammalian circadian clocks, SCN, consists of $O(10^4)$ clock cells[9]. By this reason, AT is no less important in locally coupled systems than in globally coupled systems. Besides AT, populations of coupled active and inactive oscillators can exhibit a variety of interesting behaviors, e.g. clustering in the desynchronization horn and the disorder-induced phase coherence, as we have seen above. Probably, many others will be waiting to be discovered in diverse contexts and population architectures.

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