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# Piecewise Constant Spiking Oscillators by a Pulse-coupling

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**Abstract**—This paper studies pulse-coupled piecewise constant spiking oscillators (PWCSOs) by a master-slave coupling. Since the master PWCSO exhibits chaos, the slave one exhibits chaotic response. If the parameter varies, however, the behavior of the slave PWCSO can be like periodic trajectory in the phase plane. Using the derived 2-D return map, we shows bifurcation phenomena and clarify its mechanism.

## 1. Introduction

Various coupling systems by a pulse are investigated in many researchers [1]-[7]. Analysis of these systems is important to understanding and consideration to various synchronization phenomena, bifurcation phenomena and so on. Also, applications for associative memories and image segmentations are studied, based on pulse-coupled neural networks [4]-[7].

This paper studies a pulse-coupled system of piecewise constant spiking oscillators (PWCSOs). The PWCSO has PWC vector fields and piecewise linear trajectory [8]. We consider a master-slave coupling system used two single PWCSOs. The master PWCSO exhibits chaotic attractor and spike-train output. The slave PWCSO without the coupling exhibits a stable equilibrium point. The dynamics of the pulse-coupled PWCSOs is described as the following. The states of each unit vibrate with time evolution. If the state of the master PWCSO reaches a threshold level, a spike is outputted. The spike occurs that the states of each PWCSO are reset to each base at the same time. Repeating above the manner, the pulse-coupled PWCSO outputs the spike-trains. The response of the slave PWCSO is chaotic because of chaotic spike-trains of the master one. However, if the parameters varies, the trajectory of the slave PWCSO seems to be like periodic trajectory in the phase plane. In order to analyze this ”periodicity-like trajectory”, we derive a two dimensional (2-D) return map. Using the 2-D map, we consider mechanism of the ”periodicity-like trajectory” and related bifurcation phenomena.

It should be noted that our result [10] has presented the pulse-coupled PWCSO, however it has discussed within other parameter ranges and has not discussed the ”periodicity-like trajectory” with two or more periods.

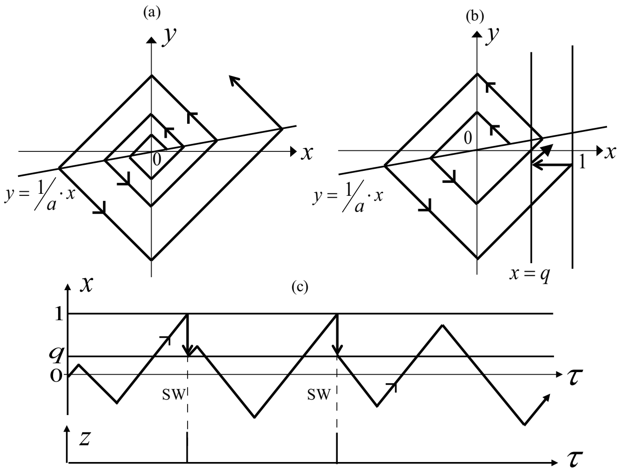


Figure 1: Basic dynamics of the PWCSO for $k = 1$, $a > 1$. (a) An unstable rect-spiral for $x < 1$, (b) A phase plane, (c) A time domain.

## 2. Piecewise Constant Spiking Oscillators

In order to consider a pulse-coupled system of the PWCSOs, as a preparation, we present two single PWCSOs [8], [9]. The first PWCSO has a trajectory with divergent vibration. The dynamics of the PWCSO with divergence vibration can be described by Equation (1).

$$\begin{cases} \dfrac{dx}{d\tau} = k\mathrm{sgn}(x - ay), \\ \dfrac{dy}{d\tau} = k\mathrm{sgn}(x), \end{cases} \quad \text{for } x(\tau) < 1,$$
$$(x(\tau+), y(\tau+)) = (q, y(\tau)) \quad \text{if } x(\tau) = 1, \tag{1}$$
$$z(\tau) = \begin{cases} 1 & \text{for } x(\tau) = 1, \\ 0 & \text{for } x(\tau) \neq 1, \end{cases}$$

where $\tau$, $x$, $y$ and $z$ are dimensionless time, state variables and an output, respectively. sgn $(\cdot)$ is the signum function:

$$\mathrm{sgn}(x) = \begin{cases} 1 & \text{for } x > 0, \\ 0 & \text{for } x = 0, \\ -1 & \text{for } x < 0. \end{cases}$$

This PWCSO has three parameters $a$, $k$ and $q$. For simplicity, we restrict

$$a > 1, \quad k = 1, \quad 0 \leq q < 1. \tag{2}$$

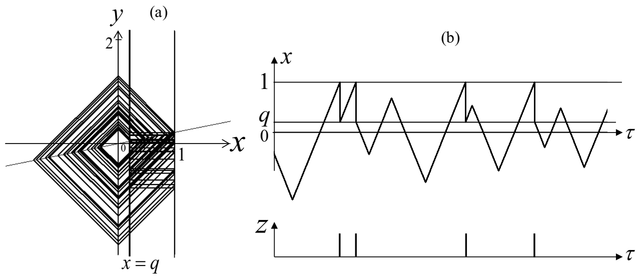


Figure 2: Typical phenomena of the PWCSO for $k = 1$, $a = 5.0$ and $q = 0.2$. (a) Chaotic attractor, (b) Time waveform and spike-trains.

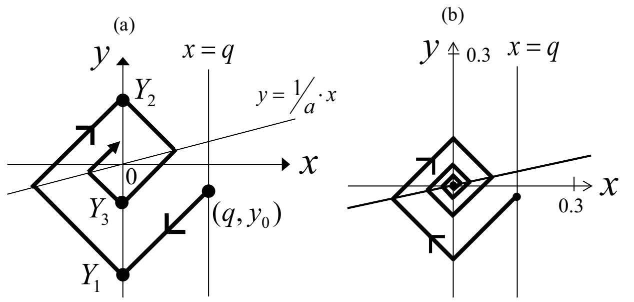


Figure 3: Typical dynamics of the PWCSO for $k = -1$. (a) A stable rect-spiral in the phase plane, (b) An equilibrium point for $a = 5.0$ and $q = 0.3$.

The vector field of this PWCSO is shown in Figure 1 (a). For $x < 1$, the trajectory can vibrate divergently around the origin and draw an unstable rectangular-spiral in the phase plane. The dynamics of the PWCSO is shown in Fig. 1 (b) and (c). Below a threshold line $x = 1$, the state $x(\tau)$ vibrates divergently. If the trajectory reaches the threshold line, the system outputs a spike $z = 1$ and then $x$ is reset to the base $q$ instantaneously. Repeating this manner, the PWCSO can output spike-trains $z(\tau)$. Fig. 2 shows typical phenomena. The PWCSO exhibits chaotic attractor and spike-trains. It should be noted that this PWCSO exhibits only chaotic behavior for parameter ranges of Eq. (2) [8].

The second PWCSO has a trajectory with convergence vibration. The dynamics of the PWCSO with convergence vibration is described by Eq. (1) where we restrict

$$a > 1, \quad k = -1, \quad 0 \le q < 1. \tag{3}$$

Since $k = -1$, the vector becomes the opposite direction as compared with divergence vibration: the trajectory moves in a "clockwise" direction. In this case as shown in Fig. 3 (a) and (b), the trajectory draws an stable rect-spiral.

Here we consider the case where the trajectory does not reach the threshold, and define some key points to present characteristics of the trajectory. As shown in Fig. 3 (a), we assume that a trajectory starts from a point $(x(0), y(0)) = (q, y_0)$ at $\tau = 0$ where $y_0 < \frac{q}{a}$. Let $(0, Y_n)$ be $n$-th intersection of the trajectory and $y$-axis where $n$ is a positive integer. $Y_1$ and $Y_n$ are given by $Y_1 = y_0 - q$ and

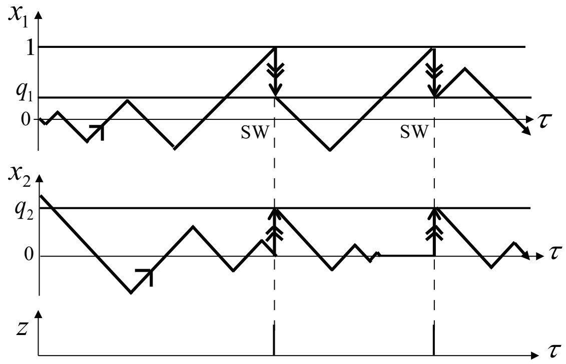


Figure 4: Typical dynamics of the pulse-coupled PWCSOs for $k_1 = 1$, $k_2 = -1$, $a_1 = a_2 > 1$ and $0 \le q_1 < 1$ and $0 \le q_2 < 1$.

$Y_n = -\alpha Y_{n-1} = -\alpha^{n-1}(y_0 - q)$ where $\alpha = \frac{a-1}{a+1}$ and $0 < \alpha < 1$ is satisfied. Let $\tau_n$ be the time at which the trajectory reaches a point $(0, Y_n)$. $\tau_n$ is given by $\tau_n = (\alpha^{n-1} - 1)aY_1 + q$. Note that this system has an important property $\tau_n > \tau_{n+1}$. Defining the time $\tau_\infty$ at which the trajectory reaches the origin: $y(\tau_\infty) = 0$, we obtain

$$\tau_\infty = -aY_1 + q = -a(y_0 - q) + q, \tag{4}$$

where $\tau_\infty > 0$ is guaranteed since $Y_1 < 0$ and Eq. (3). Note that we have described similar discussions in [9]. In this PWCSO, the trajectory can reach a stable equilibrium point (origin) at finite time as shown in Fig. 3 (b). This "reachability" characteristic can cause interesting phenomena of a pulse-coupled system in Section 3. Note that the piecewise linear system can not have the "reachability" characteristic [9].

## 3. A pulse-coupled system of the PWCSOs

In this section, we consider a pulse-coupled system of two single PWCSO. The dynamics of the pulse-coupled PWCSOs is described by following equations.

$$\begin{cases} \dot{x}_1 = k_1 \mathrm{sgn}(x_1 - a_1 y_1) \\ \dot{y}_1 = k_1 \mathrm{sgn}(x_1) \end{cases} \quad \text{for } x_1(\tau) < 1, \tag{5}$$

$$(x_1(\tau_+), y_1(\tau_+)) = (q_1, y_1(\tau)) \text{ if } x_1(\tau) = 1, \tag{6}$$

$$\begin{cases} \dot{x}_2 = k_2 \mathrm{sgn}(x_2 - a_2 y_2) \\ \dot{y}_2 = k_2 \mathrm{sgn}(x_2) \end{cases} \quad \text{for } x_1(\tau) < 1, \tag{7}$$

$$(x_2(\tau_+), y_2(\tau_+)) = (q_2, y_2(\tau)) \text{ if } x_1(\tau) = 1, \tag{8}$$

$$z(\tau) = \begin{cases} 1 & \text{for } x_1(\tau) = 1, \\ 0 & \text{for } x_1(\tau) \neq 1, \end{cases} \tag{9}$$

where " · " denotes differentiation by $\tau$. This system has six parameters $a_1$, $a_2$, $k_1$, $k_2$, $q_1$ and $q_2$. For simplicity, we assume parameters ranges as the following.

$$\begin{aligned} &a_1 = a_2 > 1, \quad k_1 = 1, \quad k_2 = -1, \\ &0 \le q_1 < 1, \quad 0 \le q_2 < 1. \end{aligned} \tag{10}$$

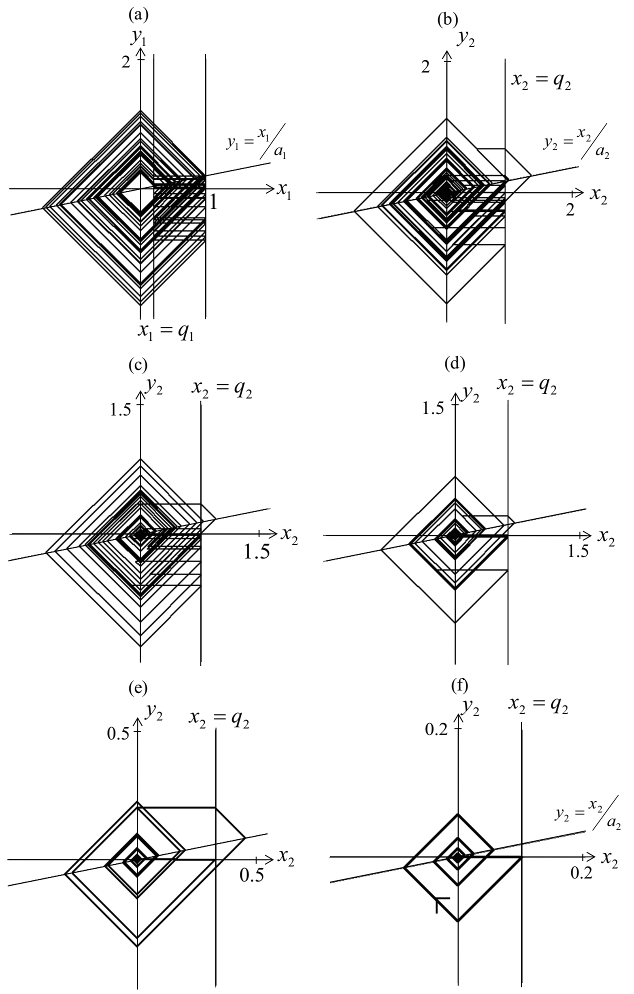


Figure 5: Typical attractors of the pulse-coupled PWCSOs for $k_1 = 1$, $k_2 = -1$, $a_1 = a_2 = 5.0$ and $q_1 = 0.2$. Fig. (a) is "Unit 1" and Figs. (b) to (f) are "Unit 2". (a) Chaos, (b) $q_2 = 0.9$, (c) $q_2 = 0.7$, (d) $q_2 = 0.6$, (e) $q_2 = 0.3$, (f) $q_2 = 0.1$.

Note that [10] have studied for $-1 < q_1 < 0$ and $-1 < q_2 < 0$. In this paper, we call the PWCSO of Eqs. (5) and (6) "Unit 1", and call the PWCSO of Eqs. (7) and (8) "Unit 2".

The dynamics of the pulse-coupled PWCSOs is shown in Fig. 4. The state $x_1(\tau)$ ( or $x_2(\tau)$ ) vibrates divergently ( or convergently ) from an initial state. If the state $x_1$ reaches the threshold $x_1 = 1$, Unit 1 outputs a spike $z(\tau)$ and the state $x_1$ is reset to the base $q_1$ instantaneously. Then, Unit 2 accepts the spike $z$ and the state $x_2$ is reset to the base $q_2$: it is a master-slave coupling. Master and slave systems are Unit 1 and Unit 2, respectively. The pulse-coupled PWCSO repeats this manner and can output the spike-trains. Fig. 5 shows typical attractors. In Fig. 5 (a), Unit 1 exhibits chaotic attractor and spike-trains. Since chaotic switching of Unit 1 occurs, the switching of Unit 2 is chaotic: The stable behavior (see Fig. 3 (b) ) is changed into chaotic behavior (see Figs. 5 (b) and (f) ).

However, the behavior of Unit 2 seems to be periodic trajectory in Fig. 5 (f). Although Unit 2 exhibits chaotic behavior because of chaotic spike-train input from Unit 1, the trajectory of Unit 2 jumps to the point $(q_2, 0)$ in the phase plane at every switching. In this paper, we call this behavior "*periodicity-like trajectory*". The periodicity-like trajectory jumps to the point $(q_2, 0)$ at every switching in the phase plane but jump (reset) timing is chaotic.

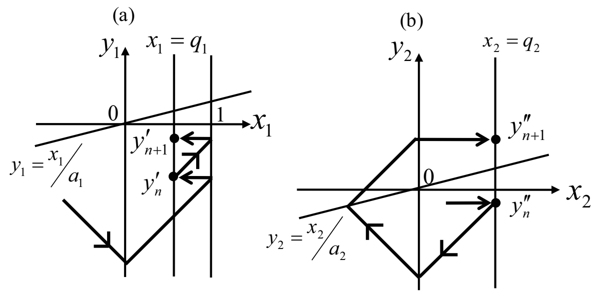


Figure 6: Definition of the 2-D return map in the phase plane. (a) Unit 1, (b) Unit 2.

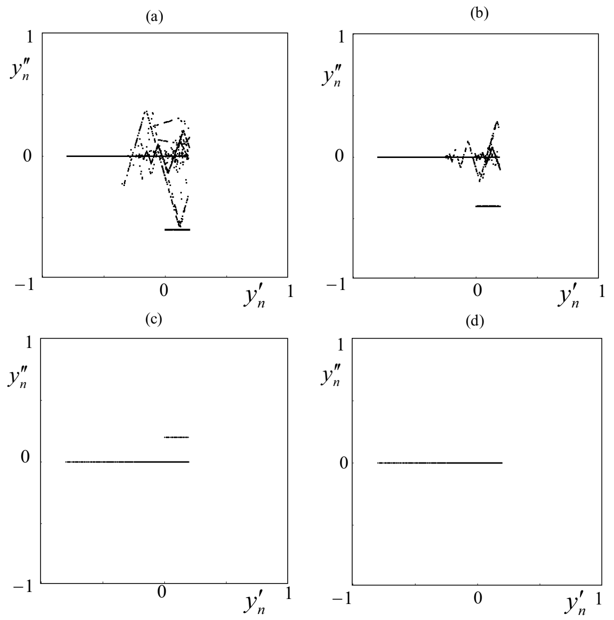


Figure 7: Typical 2-D return maps for $k_1 = 1$, $k_2 = -1$, $a_1 = a_2 = 5.0$ and $q_1 = 0.2$. (a) $q_2 = 0.7$, (b) $q_2 = 0.6$, (c) $q_2 = 0.3$, (d) $q_2 = 0.1$. Fig. (a) to (d) correspond to Fig. 5 (c) to (f), respectively.

For simplicity, we fix the parameters $a_1 = a_2 = 5.0$, $k_1 = 1$, $k_2 = -1$ hereafter. In order to analyze the phenomena, we derive a two-dimensional return map. We define a domain $D_q$ to derive the 2-D return map: $D_q = \{x_1, y_1, x_2, y_2 \mid x_1 = q_1, x_2 = q_2\}$ and let any point in $D_q$ be represented by the states of $y_1$ and $y_2$. Fig. 6 shows key objects for the return map. Let us consider the case where the switching generates at $\tau = 0$ and the trajectories of each unit jump to each base $q_1$ and $q_2$, respectively. We represent their states as the following: $y_1(0) = y'_n$ and $y_2(0) = y''_n$. As shown in Fig. 6, the trajectories of each unit evolve with

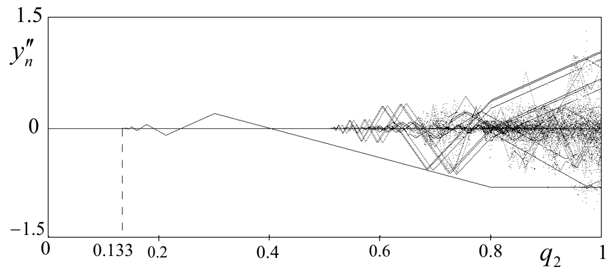


Figure 8: A bifurcation diagram of Unit 2 for $k_1 = 1$, $k_2 = -1$, $a_1 = a_2 = 5.0$ and $q_1 = 0.2$.

time. When the trajectory of Unit 1 reaches the threshold at finite positive time $\tau = \tau_1$, each trajectory jumps to each point $(q_1, y'_{n+1})$ and $(q_2, y''_{n+1})$ : let $y_1(\tau_1) = y'_{n+1}$ and $y_2(\tau_1) = y''_{n+1}$. Since the trajectory starting from $D_q$ returns to itself, the 2-D return map $F$ can be described as the following.

$$F : D_q \to D_q, \quad (y'_n, y''_n) \mapsto (y'_{n+1}, y''_{n+1}), \qquad (11)$$

where $n$ is a non-negative integer. Typical return maps are shown in Fig. 7. Since the trajectory of Unit 1 is piecewise linear, $y'_1$ and a switching interval can be calculated rigorously if an initial state $y'_0$ is given [8]. Therefore, the switching interval and the state $y''_0$ are given, $y''_1$ can be also calculated rigorously. As shown in Fig. 7 (d), we can see that a sequence $\{y''_n\}$ is all zero after transient process when Unit 2 exhibits the periodicity-like trajectory. That is, the jumping point of the trajectory of Unit 2 in the phase plane is $(q_2, 0)$ at every switching.

Fig. 8 shows a bifurcation diagram of Unit 2. When $q_2$ approaches 0, Unit 2 tends to exhibit the periodicity-like trajectory. Based on elemental calculation, we obtain that the periodicity-like trajectory occurs if Eq. (12) is satisfied.

$$q_2 < \frac{1 - q_1}{1 + a_2}. \qquad (12)$$

We consider the case where a trajectory of Unit 2 starts from a point $(q_2, 0)$ at $\tau = 0$ in the phase plane. From Eq. (4), $\tau_\infty = (a_2 + 1)q_2$ is calculated for Unit 2. On the other hands, a minimal switching interval of Unit 1 is a case where a sequent switching generates as shown in Fig. 6 (a): this switching interval is $(1 - q_1)$. Consequently, if $(a_2 + 1)q_2 < 1 - q_1$ is satisfied, the trajectory starting from $(q_2, 0)$ can reach the origin at every switching. $q_2 = \frac{1-q_1}{1+a_2} \approx 0.133$ is calculated for the parameters of Fig. 8.

For $q_2 = 0.3$, the attractor of Unit 2 seems to be like periodic attractor with period 2 (see Fig. 5 (e) ). In this case, Eq. (12) is not satisfied and the trajectory does not jump to $(q_2, 0)$ at every switching. When the sequent switching of Unit 1 generates, the trajectory of Unit 2 can not reach the origin, however other cases generate reaching of the trajectory. The 2-D return map is shown in Fig. 7 (c). That is, the trajectory of Unit 2 jumps to two points in the phase plane. We could say that the periodicity-like trajectory with period $n$ exists in wider parameter ranges. It should be noted that this behavior is also chaotic response because of chaotic input of Unit 1.

## 4. Conclusions

We have studied the dynamics of the pulse-coupled PWCSOs. In the master-slave coupling, the pulse-coupled PWCSO exhibits various phenomena including the ”periodicity-like trajectory”. Using the 2-D return map, we have studied occurrence mechanism of the periodicity-like trajectory. Future problems include a more detailed analysis of the periodicity-like trajectory with period $n$ and bifurcation phenomena in wider parameter ranges.

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