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Cusps of periodic orbits in Chua's equation

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Abstract—We study the Hopf bifurcation of the nontrivial equilibria and its degeneracies in Chua's equation. The existence of cusp bifurcations of periodic orbits is proved. Numerical continuation results are in perfect agreement with theoretical ones.

1. Introduction

Chua's equation models one of the simplest electronic circuits that exhibits a wide range of complex dynamical behaviors. Let us consider the case of a cubic nonlinearity, whose equations are derived in [1]:

$$\begin{aligned} \dot{x} &= \alpha(y - ax^3 - cx), \\ \dot{y} &= x - y + z, \\ \dot{z} &= -\beta y - \gamma z, \end{aligned} \tag{1}$$

with $a = \pm 1$, $\alpha \neq 0$, $\beta \neq 0$ and $\gamma, c \in \mathbb{R}$. Note that for $\alpha = 0$ system (1) is linear and if $\beta = 0$ it is uncoupled. In this system a parameter γ is included in order to take into account small resistive effects in the inductance.

System (1) is $\mathbb{Z}_2$-symmetric, that is, it is invariant with respect to the change $(x, y, z) \to (-x, -y, -z)$. Thus, the origin is always an equilibrium point. Moreover, for $a(\gamma/(\beta + \gamma) - c) > 0$ two nontrivial equilibria exist at $(x_0, ax_0^3 + cx_0, ax_0^3 + (c-1)x_0)$, with $ax_0^2 = \gamma/(\beta + \gamma) - c$. Generically, on surface $c(\beta + \gamma) = \gamma$ a pitchfork bifurcation of equilibria occurs.

The analysis of the pitchfork bifurcation for Chua's equation is performed in [2]. In this work, authors also consider Takens-Bogdanov bifurcation of the origin and its nonlinear degeneracies when parameter $\gamma = 0$. The case $\gamma \neq 0$ has been studied in [3], where the triple-zero bifurcation is also examined. The theoretical analysis of the Hopf-pitchfork bifurcation as well as its nonlinear degeneracies can be found in [4].

To complete the study of all the local bifurcations exhibited by system (1) we consider here the Hopf bifurcation of the nontrivial equilibria and its degeneracies when $\gamma > 0$. The case $\gamma = 0$ has been recently considered in [5]. The presence of cusps of saddle-node bifurcations of periodic orbits is proved by our analysis and thus, the coexistence of three-periodic orbits and hysteretic phenomena are guaranteed.

2. Hopf Bifurcation of the Nontrivial Equilibria

Due to the Z_2-symmetry Chua's equation exhibits, it is enough to analyse the Hopf bifurcation of one of the two nontrivial equilibria, namely (x_0, y_0, z_0). The non-trivial equilibria exist when the condition

$$a\left(\frac{\gamma}{\beta+\gamma} - c\right) > 0$$

is fulfilled.

By means of the change $x = u + x_0$, $y = v + y_0$, $z = w + z_0$, the nontrivial equilibrium is translated to the origin, and system (1) is transformed into

$$\begin{aligned} \dot{u} &= \left(2\alpha c - \tfrac{3\alpha\gamma}{\beta+\gamma}\right)u + \alpha v - 3x_0\alpha a u^2 - a\alpha u^3, \\ \dot{v} &= u - v + w, \\ \dot{w} &= -\beta v - \gamma w. \end{aligned} \tag{2}$$

The characteristic polynomial of the linearization matrix at the nontrivial equilibria is $P(\lambda) = \lambda^3 + p_1\lambda^2 + p_2\lambda + p_3$ with

$$\begin{aligned} p_1 &= \frac{3\alpha\gamma}{\beta+\gamma} + (1+\gamma) - 2\alpha c, \\ p_2 &= \beta + \gamma - \alpha - 2\alpha c(1+\gamma) + \frac{3\alpha\gamma(1+\gamma)}{\beta+\gamma}, \\ p_3 &= 2\alpha\left[c(\beta+\gamma) - \gamma\right]. \end{aligned} \tag{3}$$

The Hopf bifurcation is obtained for $p_3 = p_2 p_1$ where $p_1 \neq 0$, $p_2 > 0$.

The nontrivial equilibria undergo a Hopf bifurcation when

$$\mathbf{h} \equiv \begin{cases} [2\alpha c(1+\gamma) + \beta_0]^2 \\ \qquad -\dfrac{1}{4}\left[\alpha_0 - 4(1+\gamma)^2(\beta+\gamma)\right] = 0, \\ \dfrac{3\alpha\gamma}{\beta+\gamma} + (1+\gamma) - 2\alpha c \neq 0, \\ \beta + \gamma - \alpha - 2\alpha c(1+\gamma) + \dfrac{3\alpha\gamma(1+\gamma)}{\beta+\gamma} > 0, \end{cases}$$

where

$$\begin{aligned} \beta_0 &= -\frac{3\alpha\gamma(1+\gamma)}{\beta+\gamma} + \frac{\alpha - (1+\gamma)^2}{2}, \\ \alpha_0 &= (1 + \alpha - \gamma^2)^2 + 4\gamma(1+\gamma)^2. \end{aligned}$$

Let us denote, for the sake of simplicity, $\tilde{\beta} = \beta + \gamma$. For $\alpha \neq 0$ and $p_1 \neq (1 + 2\gamma)$ Eqs. (3) define explicitly $\alpha, \tilde{\beta}, c$ as functions of p_1, p_2, p_3 since

$$\begin{array}{rcl} \dfrac{\partial(p_1, p_2, p_3)}{\partial(\alpha, \tilde{\beta}, c)} & = & 2\alpha\left(\dfrac{3\alpha\gamma}{\tilde{\beta}} - 2\alpha c - \gamma\right) \\ & = & -2\alpha[(1 + 2\gamma) - p_1] \neq 0. \end{array}$$

Denoting $p_1 = -\lambda_h$, taking into account that $p_1 p_2 = p_3$ and $p_2 = \omega_0^2 > 0$ and solving Eqs. (3) for $\alpha, \tilde{\beta}, c$, the Hopf manifold of the nontrivial equilibria reads

$$\mathbf{h} \equiv \left\{ \begin{array}{l} \alpha = -\dfrac{(1 + \gamma)[(1 + \gamma + \lambda_h)^2 + w_0^2]}{1 + 2\gamma + \lambda_h}, \\ \tilde{\beta} = \dfrac{(\gamma + \lambda_h)w_0^2 + \gamma(1 + \gamma)(1 + \gamma + \lambda_h)}{1 + 2\gamma + \lambda_h}, \\ c = \dfrac{3\gamma}{2\tilde{\beta}} + \dfrac{1 + \gamma + \lambda_h}{2\alpha}, \\ \omega_0^2 > 0, \ \ \lambda_h \neq 0. \end{array} \right.$$

Observe that if $\lambda_h = -1 - 2\gamma$ and $(\alpha, \beta, c) \in \mathbf{h}$ then $\gamma = -1$.

In order to analyse the possible degeneracies of the Hopf bifurcation we consider system (2) with parameters evaluated at their critical values and perform the linear transformation given by the matrix

$$\left(\begin{array}{ccc} \omega_0(1 + \gamma) & (1 + \gamma)\eta_1 & \eta_1^2 + \omega_0^2 \\ 0 & \eta_2 & \eta_2 \\ \omega_0(\gamma + \lambda_h) & -\gamma(\gamma + \lambda_h) & -(\gamma^2 + \omega_0^2) \end{array} \right),$$

where $\eta_1 = 1 + \gamma + \lambda_h$ and $\eta_2 = 1 + 2\gamma + \lambda_h$.

Thus we obtain a new system whose linearization matrix is in canonical form

$$\left(\begin{array}{ccc} 0 & -\omega_0 & 0 \\ \omega_0 & 0 & 0 \\ 0 & 0 & \lambda_h \end{array} \right).$$

Using the recursive algorithm given in [6], the fifth-order normal form for the reduced system on the center manifold in polar coordinates is

$$\begin{array}{rcl} \dot{r} & = & a_1 r^3 + a_2 r^5, \\ \dot{\theta} & = & \omega_0 + b_1 r^2 + b_2 r^4, \end{array}$$

where

$$\begin{array}{l} a_1 = \dfrac{3a^2 x_0^2 (1 + \gamma)^5 \left[\eta_1^2 + \omega_0^2\right]^3 a_{11}}{2\lambda_h \omega_0^2 (\lambda_h^2 + 4\omega_0^2)(\lambda_h^2 + \omega_0^2)\eta_2^4}, \\ a_2 = \dfrac{3a^4 x_0^4 (1 + \gamma)^9 \left[\eta_1^2 + \omega_0^2\right]^6 a_{21}}{2\lambda_h^3 \omega_0^6 \eta_2^8 (\lambda_h^2 + \omega_0^2)^3 (\lambda_h^2 + 9\omega_0^2)(\lambda_h^2 + 4\omega_0^2)^3}, \end{array} \quad (4)$$

with

$$\begin{array}{rcl} a_{11} & = & -10(\gamma + \lambda_h)(\omega_0^2)^3 + [(12\gamma - 5)\lambda_h^2 \\ & & +(12\gamma^2 - 11\gamma - 6)\lambda_h - 10\gamma(\gamma + 1)](\omega_0^2)^2 \\ & & +[4\lambda_h^5 + 5(5\gamma + 2)\lambda_h^4 + (58\gamma^2 + 56\gamma + 9)\lambda_h^3 \\ & & +(\gamma + 1)(73\gamma^2 + 43\gamma + 3)\lambda_h^2 \\ & & +\gamma(46\gamma + 13)(\gamma + 1)^2\lambda_h + 10\gamma^2(\gamma + 1)^3]\omega_0^2 \\ & & +\gamma\lambda_h^2(\gamma + 1)(\lambda_h + \gamma + 1)[\lambda_h^2 \\ & & +(3\gamma + 1)\lambda_h + \gamma(\gamma + 1)], \end{array}$$

and a_{21} a polynomial (with tenth degree in ω_0^2 provided that $\gamma + \lambda_h \neq 0$) whose coefficients are polynomials in γ and λ_h, omitted for the sake of brevity. The azymutal coefficients b_1 and b_2 are not involved in the study of this bifurcation.

The analysis of the stability and the degeneracies of the Hopf bifurcation of the nontrivial equilibria reduces to the study of the signs and zeroes of coefficients a_1 and a_2 obtained in Eq. (4).

Achieving this study we get the following result [7].

Theorem. Fixed $\gamma > 0$, Chua's equation undergoes a Hopf bifurcation of the nontrivial equilibria for $(\alpha, \beta, c) \in \mathbf{h}$. Two different degenerate cases appear.

i) If $a = +1$, there are two unbounded curves of degeneracies of codimension two. The first one emerges from the codimension-three point $(\alpha_{\mathbf{TB1D}}, \beta_{\mathbf{TB1D}}, c_{\mathbf{TB1D}})$

$$\left\{ \begin{array}{l} \alpha_{\mathbf{TB1D}} = -\dfrac{(1 + \gamma)^3}{2\gamma - \sqrt{1 + 2\gamma + 5\gamma^2}}, \\ \beta_{\mathbf{TB1D}} = \dfrac{2\gamma(1 + \gamma)^2}{(1 + 3\gamma) - \sqrt{1 + 2\gamma + 5\gamma^2}} - \gamma, \\ c_{\mathbf{TB1D}} = \dfrac{(1 + 3\gamma) - \sqrt{1 + 2\gamma + 5\gamma^2}}{2(1 + \gamma)^2}, \end{array} \right.$$

(where a degenerate Takens-Bogdanov bifurcation of homoclinic type exists) and tends to a point at infinity in the space of parameters, and it exists for $\alpha > 0, \beta > 0$ and $c < 1$. The second one starts from the codimension three point $(\alpha_{\mathbf{HPD}}, \beta_{\mathbf{HPD}}, c_{\mathbf{HPD}})$

$$(\alpha_{\mathbf{HPD}}, \beta_{\mathbf{HPD}}, c_{\mathbf{HPD}}) = \left(-(1 + \gamma)^2, \gamma^2, \frac{1}{1 + \gamma}\right),$$

(where a degenerate Hopf-pitchfork bifurcation occurs) and tends to a point at infinity, and it exists for $\alpha < 0$, $\beta > 0$ and $c < 1$.

ii) If $a = -1$, there are two curves of degeneracies of codimension two. One of them is bounded and the other one is unbounded. The first curve joins the codimension-three points $(\alpha_{\mathbf{TB2D}}, \beta_{\mathbf{TB2D}}, c_{\mathbf{TB2D}})$

$$\left\{ \begin{array}{l} \alpha_{\mathbf{TB2D}} = -\dfrac{(1 + \gamma)^3}{2\gamma + \sqrt{1 + 2\gamma + 5\gamma^2}}, \\ \beta_{\mathbf{TB2D}} = \dfrac{2\gamma(1 + \gamma)^2}{(1 + 3\gamma) + \sqrt{1 + 2\gamma + 5\gamma^2}} - \gamma, \\ c_{\mathbf{TB2D}} = \dfrac{(1 + 3\gamma) + \sqrt{1 + 2\gamma + 5\gamma^2}}{2(1 + \gamma)^2}, \end{array} \right.$$

(where a degenerate Takens-Bogdanov bifurcation of homoclinic type exists) and $(\alpha_{\mathbf{HPD}}, \beta_{\mathbf{HPD}}, c_{\mathbf{HPD}})$ and it exists for $\alpha < 0$, $\beta > 0$ and $0 < c < 3/2$. In addition, a codimension-three point appears on the quoted curve (this new degeneracy leads to the presence of cusps of saddle-node bifurcations of periodic orbits). The second curve joins two points at infinity and it exists for $\alpha < 0, \beta < 0$ and $c > 1$.

We note that the Hopf bifurcations of the origin and its degeneracies in Chua's equation are also considered in [7].

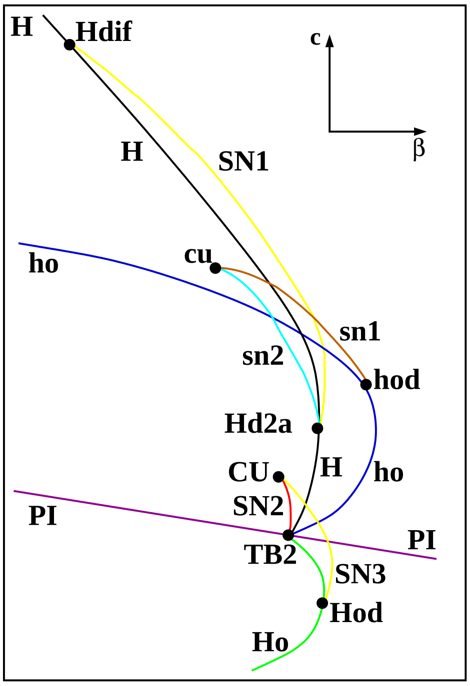

Figure 1: Qualitative partial bifurcation set for $\alpha = -1.085$, $\gamma = 0.3$ and $a = -1$.

3. Numerical Results

In this section we take advantage of the analytical results obtained above. In our numerical study of Chua's equation (1), performed basically with AUTO [8], we will take β, c and α as bifurcation parameters and we will fix the other two, namely $a = -1$, $\gamma = 0.3 > 0$, in accordance with the previous papers [3, 4] that the present work complements in certain aspects. For the values of a and γ fixed, a degenerate Takens-Bogdanov bifurcation of homoclinic type **TB2D** takes place at the critical values $\alpha \approx -1.08132$, $\beta \approx 0.00434$ and $c \approx 0.98573$.

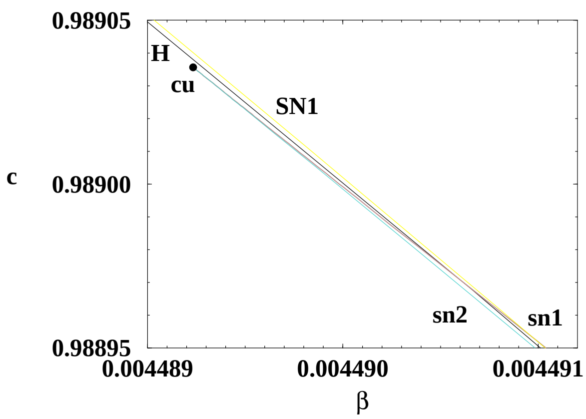

Figure 2: Zoom of Fig. 1 in a neighborhood of the cusp point **cu**.

In Fig. 1 we present a qualitative partial bifurcation set for $\alpha = -1.085$. We observe that a point of Takens-Bogdanov bifurcation of the origin **TB2** exists on the curve where the pitchfork bifurcation of the origin **PI** occurs. This locus **PI** is given by

$$c = \frac{\gamma}{\beta + \gamma}.$$

From the codimension-two point **TB2** four curves emerge: **Ho**, corresponding to a subcritical Hopf bifurcation of the origin (where a symmetric periodic orbit is born); **ho**, a curve where the nontrivial equilibria undergo a supercritical Hopf bifurcation (and then a pair of asymmetric periodic orbits is born); **H**, a curve of homoclinic connections of the origin and **SN2** corresponding to saddle-node bifurcations of symmetric periodic orbits. The homoclinic connection **H** that emerged attractive from **TB2** changes to repulsive when it crosses the degeneration point **Hd2a** (this degeneracy occurs when $\delta = |\rho/\lambda| = 1$, where $(\rho \pm i\omega, \lambda)$ are the eigenvalues of the origin). Consequently, two saddle-node curves emerge from **Hd2a** entering in the repulsive homoclinic region, one of symmetric periodic orbits, **SN1**, and the other of asymmetric periodic orbits, **sn2**. This last curve collapses in a cusp, **cu**, with the saddle-node curve of asymmetric periodic orbits, **sn1**, emerged from the degeneration on the Hopf bifurcation of the nontrivial equilibria, **hod**. Moreover, the saddle-node curves of symmetric periodic orbits, **SN2** and **SN3** (emerged from the degeneration on the Hopf bifurcation of the origin, **Hod**), collapse in a cusp of saddle-node of symmetric periodic orbits **CU**. The curve **SN1** ends at a new codimension-two point, **Hdif**, placed on **H**, which corresponds to a degeneration in the homoclinic connection known as *inclination-flip* or *critical twist* [9]. In Fig. 2 we show a detail of the region where a codimension-two point, **cu** (cusp of asymmetric periodic orbits) appears.

To analyze these new codimension-two bifurcations we continue, in the (α, β, c)-parameter space, the codimension-two curves. In this way, the projection of such curves onto the (α, c) plane is shown in Fig. 3.

From the codimension-three point **TB2D**, that corresponds to a degenerate Takens-Bogdanov bifurcation of homoclinic type [10], four curves of codimension-two bifurcations emanate: a degenerate Hopf bifurcation of the origin **Hod**, a degenerate Hopf bifurcation of the nontrivial equilibria **hod** (studied in Sec. 2), a cusp of saddle-node bifurcations of periodic orbits **CU** and a degenerate homoclinic connection **Hd2a**. The curve **hod** ends outside the range of the parameters shown in Fig. 3, at the point **HPD** ($\alpha = -1.69$, $\beta = 0.09$, $c \approx 0.76923$) where a degenerate Hopf-pitchfork bifurcation of the origin occurs. This codimension-three bifurcation was analyzed in [4]. Also in Fig. 3 the codimension-three point **ThoD** (where a second-order degeneracy of the Hopf bifurcation of the nontrivial equilibria occurs) appears on the curve **hod**, for $\alpha \approx -1.08202$, $\beta \approx 0.00436$ and $c \approx 0.98768$. From such

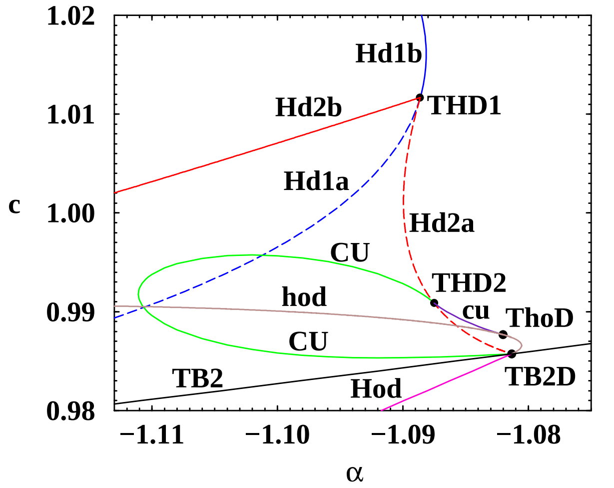

Figure 3: A partial bifurcation set in a neighborhood of the degenerate Takens-Bogdanov point **TB2D**, for $\gamma = 0.3$ and $a = -1$.

a point the curve **cu**, that ends in other three-codimension point **THD2**, emerges. This last point, for $\alpha \approx -1.0875$, $\beta \approx 0.00457$ and $c \approx 0.9909$, corresponds to a degenerate homoclinic connection verifying simultaneously that $\delta = 1$ (the origin is resonant) and $\bar{a} = 1$, being $\bar{a}$ the coefficient of the Poincaré next return map: $x \to \varepsilon + \bar{a}|x|^{\delta}$. The cases $\bar{a} > 1$ and $\bar{a} < 1$ have been studied in [11] and the codimension-three degeneracy in [12]. These results agree with the analytical ones obtained in Sec. 2 for the case $a = -1$.

4. Conclusion

In this paper we have considered the Hopf bifurcation of the nontrivial equilibria in Chua's equation. We have proved that a codimension-three degeneracy appears, that is, a cusp of saddle-node bifurcations of periodic orbits that can give rise to hysteretic phenomena. Numerical results presented are in perfect agreement with the analytical ones obtained.

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