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Measuring the Complexity of Time Series Using Ordinal Partition Networks

Michael McCullough†, Michael Small‡ and Herbert Ho-Ching Iu†

†School of Electrical and Electronic Engineering, The University of Western Australia
Crawley WA 6009, Australia
‡School of Mathematics and Statistics, The University of Western Australia
Crawley WA 6009, Australia
Email: michael.mccullough@research.uwa.edu.au

Abstract—Recent investigations of ordinal partition networks show potential for the method as a tool for analysing nonlinear time series from continuous systems. In this paper we demonstrate how interpolation can be used to minimise node aliasing such that ordinal partition networks more accurately capture dynamics from discrete time sampled data. We then show how the transitional complexity of a time series can be quantified using a weighted average of node entropy and investigate this measure as applied to the Rössler system.

1. Introduction

Various methods for the analysis of dynamical systems from time series using complex networks have garnered attention in recent years [1]. One of the latest concepts to emerge is that of ordinal partition networks, as first proposed in [2] and subsequently generalised in [3]. This method is an extension of the symbolic mapping process used in permutation entropy [4, 5] and is of notable potential for several reasons. Firstly, it has been shown that the existence of different ordinal patterns in time series from a specific system is a property directly related to the dynamics of that system [6]. Ordinal patterns are also relatively robust to noise yet still able to capture small scale amplitude differences, unlike coarse graining methods. A recent investigation of ordinal partition networks has shown that ordinal patterns can be interpreted as a partition of the embedding phase space and hence, that the networks are a Markov model of the underlying dynamics of a time series [3]. This same study also demonstrated that simple measures on these networks, such as mean node degree and node degree variance, can track dynamical changes in continuous systems, both simulated and experimental.

The purpose of this paper is twofold. Firstly, we discuss the use of interpolation to reduce node aliasing in ordinal partition networks (Section 2.2). Secondly we introduce the concept of node entropy [7, 8] as applied to these networks (Section 2.3), where it becomes an intuitive measure of time series complexity and, as will be shown, provides effective tracking of dynamical change in a continuous Rössler system (Section 3).

2. Method

2.1. The Ordinal Partition Network Algorithm

As defined in [3], the time series $x = \{x_1, x_2, x_3, ..., x_n\}$ is embedded with lag τ and dimension D to form the sequence of state vectors $v_t = (x_t, x_{t+\tau}, x_{t+2\tau}, ...x_{t+(D-1)\tau})$. Each state vector is then mapped to a symbol — an ordinal partition element $s_i = (\pi_1, \pi_2, ..., \pi_D)$ where $\pi_j \in \{1, 2, ..., D\}$ and $\pi_j \neq \pi_k \iff j \neq k$ such that $\pi_j < \pi_k \iff x_j > x_k \; \forall x_j, x_k \in v_t$. If $x_j = x_k$ then rank is assigned based on order of appearance in v_t. Each unique s_i is mapped to a node in the network represented by adjacency matrix A. Weighted and directed edges are allocated between nodes based on temporal succession in the symbolic sequence $S = \{s_i^1, s_i^2, s_i^3, ..., s_i^{(n-D+1)}\}$ that corresponds to the sequence of state vectors $v = \{v_1, v_2, v_3, ..., v_{(n-D+1)}\}$. The edge weight $a_{i,j}$ equals the number of times that the pair $\{s_i^n, s_j^{n+1}\}$ occurs in S.

With regards to parameter selection, the embedding lag τ should be selected to optimise the embedding using, for example, the autocorrelation method, mutual information or similar. The embedding dimension D should be selected large enough to eliminate or, in a practical sense, to minimise the occurrence of degeneracies in the network. We define a degeneracy in an ordinal partition network to be a pair of state vectors $\{v_j, v_k\}$ corresponding to two independent segments of trajectory (i.e. not close in time) that are mapped to the same symbol by virtue of a non-ideal partition. Degeneracies manifest as shortcuts in the network between states that are not be close with respect to the dynamics of the underlying system and, hence, corrupt the model. Presently there is no explicit algorithm for determining an optimal D, however, a suitable value can usually be selected based on visual inspection of a plot of an appropriate network statistic over a sufficiently large range of range of D.

2.2. Node Aliasing

Consider the trivial case of a noiseless periodic time series x that is sampled at a frequency $\omega \neq b\omega_0, \forall b \in \mathbb{N}$ where ω_0 is the fundamental frequency of oscillation of x. The possibility then arises that for the temporally adjacent pair of embedded state vectors $\{v_j, v_{j+1}\}, \exists v_k$ such that v_k

is an intermediate point on the trajectory between v_j and v_{j+1}. Depending on the sampling rate, the choice of embedding parameters and the particular nature of the trajectory in question, several connectivity scenarios can occur in the network that do not reflect the true behaviour of the underlying dynamical system. For example, if $v_j \to s_1$, $v_{j+1} \to s_3$ we already know that there will be a directed edge for $\{s_1, s_3\}$. Consider also $v_k \to s_2$, and v_{k-1} sufficiently close to v_j such that $v_{k-1} \to s_1$. The resulting subgraph $\{s_1, s_2, s_3\} \subset A$ is then:

$$\begin{array}{cc} & \begin{array}{ccc} s_1 & s_2 & s_3 \end{array} \\ \begin{array}{c} s_1 \\ s_2 \end{array} & \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \end{array}$$

which implies that, from the perspective of node s_1, nodes s_2 and s_3 appear to be indistinguishable as destinations states , hence we call this phenomenon *node aliasing*. This kind of transitional uncertainty cannot occur in a strictly periodic system, for which we would instead expect the following subgraph:

$$\begin{array}{cc} & \begin{array}{ccc} s_1 & s_2 & s_3 \end{array} \\ \begin{array}{c} s_1 \\ s_2 \end{array} & \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array}$$

It is therefore necessary to select a sufficiently high sample rate with respect to the data, but also with respect to the embedding dimension D, because that parameter governs the effective size of each ordinal partition in the embedding phase space [3]. In practice sampling rates are limited, however, it should be possible to minimise aliasing if we assume a smooth trajectory between sampled points and interpolate the data to a sufficiently high resolution. Figure 1 shows networks generated using a discrete sampled time series from a period-3 Rössler system using both the original data, where significant evidence of node aliasing is clearly observable in the network (Figures 1(a) and (c)), and interpolated data, for which node aliasing has been almost completely eliminated (Figures 1(b) and (d)).

In the chaotic case, node aliasing causes the network to collapse on itself (Figure 2(b)). After interpolation, the time series maps to a network that is far more intuitive from a qualitative perspective, and for which it is possible to observe subgraphs that correspond to the stretching and folding regions of the attractor (Figure 2(c)).

2.3. Weighted Average Node Entropy

We define node entropy as in [7]. First we compute the stochastic matrix P with entries $p_{i,j}$ estimated from the weighted adjacency matrix A:

$$p_{i,j} = \frac{a_{i,j}}{\sum\limits_{k \in \mathcal{N}(i)} a_{i,k}} \,, \qquad (1)$$

where $\mathcal{N}(i)$ denotes the set of out-connected neighbours of node s_i. The node entropy of s_i is the Shannon entropy with respect to the i-th row of P given by:

$$\mu_i = -\frac{1}{\log_2 k_i^{out}} \sum_{j \in \mathcal{N}(i)} p_{i,j} \log_2 p_{i,j} \,. \qquad (2)$$

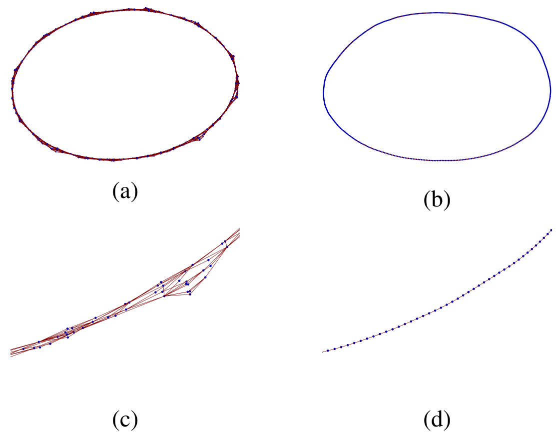

Figure 1: (a) The ordinal partition network generated from the x-component of a period-3 Rössler system originally sampled at intervals of 0.2 and (b) as generated from the same time series that has been interpolated using a cubic spline with 200 evenly space points for each original datum. Subfigures (c) and (d) show magnified portions of the network structures of (a) and (b) respectively. The embedding lag is $\tau = 8$ for (a) and (c), and $\tau = 1600$ for (b) and (d). The embedding dimension is $D = 14$ in each case.

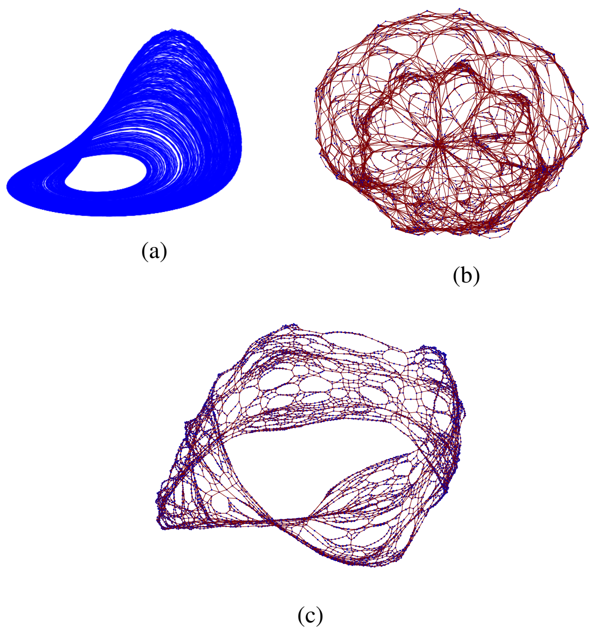

Figure 2: (a) A chaotic Rössler attractor and corresponding ordinal partition networks generated from the x-component of the system (b) as originally sampled at intervals of 0.2, and (c) interpolated using a cubic spline with 200 evenly space points for each original datum. The embdedding lag is $\tau = 8$ for (b), and $\tau = 1600$ for (c). The embedding dimension for both (b) and (c) is $D = 14$.

We use the base 2 logarithm and therefore μ_i is the average amount of information generated by s_i in bits. Note that a normalisation factor of $1/\log_2 k_i^{out}$ is applied, where k_i^{out} is the out-degree of s_i, such that $0 \le \mu_i \le 1$. If the network contains a node with $k_i^{out} = 0$ we assign node entropy $\mu_i = 0$. Strictly speaking, node entropy is undefined for $k_i^{out} = 0$, however, ordinal partition networks can have at most one such node, $s_i^{(n-D+1)}$, being the final node in the symbolic sequence.

We then take a weighted average over the network using the same principle as in [8], that is to weight each μ_i by the corresponding probability of s_i from the stationary distribution, which we estimate from A:

$$\bar{\mu}_W = \sum_i \mu_i \frac{\sum_j a_{i,j}}{\sum_{k,j} a_{k,j}} . \tag{3}$$

Therefore, weighted average node entropy measures the dynamical complexity of the partitioned time series or, alternatively, describes the expected value of transitional complexity on the reconstructed attractor. By contrast, permutation entropy and related metrics only measure global complexity based only the stationary distribution of the time series and, hence, discard temporal information.

3. Results and Discussion

We now apply the ordinal partition network method and the weighted average node entropy metric to the Rössler system for a range of the bifurcation parameter. Figure 3 shows convergence of $\bar{\mu}_W$ as the interpolation resolution is increased. From this we infer the convergence of the structure of the network itself, resulting from the elimination of node aliasing.

We then refer to a plot of $\bar{\mu}_W$ against the embedding dimension based on a selection of sufficiently interpolated time series characterised by a range of periodic and chaotic dynamics (Figure 4). Observe that the growth of $\bar{\mu}_W$ is approximately linear when $D \ge 14$ for all of the time series considered. The linear growth range indicates the elimination of the majority of degeneracies in the network, so we select $D = 14$ as an appropriate parameter value for the ensuing bifurcation analysis. Note that the period-12 time series is the last to reach linear growth w.r.t. D. This is due to the closeness of the trajectories in phase space which make networks from this particular limit cycle prone to degeneracies.

Figures 5, 6 and 7 show respectively the Rössler bifurcation spectrum, and the largest Lyapunov exponent λ_1 and weighted average node entropy $\bar{\mu}_W$ plotted against the bifurcation parameter. We observe that the complexity of the system, as quantified by $\bar{\mu}_W$, tracks relatively with λ_1, specifically with regards to the identification of periodic windows. As expected, periodic behaviour is characterised by $\bar{\mu}_W$ close to zero. Furthermore, it would be feasible to

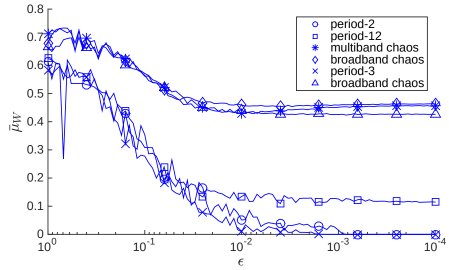

Figure 3: The weighted average node entropy $\bar{\mu}_W$ of ordinal partition networks for a selection of periodic and chaotic Rössler time series (see legend) originally sampled at intervals of 0.2 and then interpolated using a cubic spline with increasing interpolation resolution as specified by the parameter ϵ. The value $1/\epsilon$ is the number of evenly spaced interpolated points generated for each original datum. A suitable embedding lag for the original data (not interpolated) is determined to be $\tau = 8$ by the method of finding the first zero of the autocorrelation of the time series. For this and all other figures we have set the $\tau = \mathrm{nint}(8/\epsilon)$ so that the embedding parameters remain approximately equivalent with respect to the original sampling frequency for all ϵ. The embedding dimension is constant at $D = 14$.

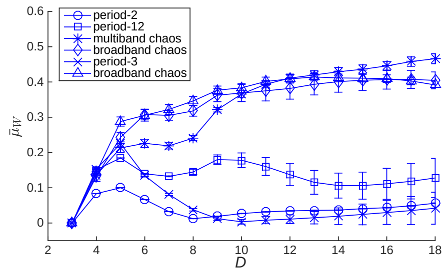

Figure 4: The mean of the weighted average node entropy $\bar{\mu}_W$ of ordinal partition networks for 100 realisations of time series from selected periodic and chaotic regimes of the Rössler system (see legend) that have been interpolated using a cubic spline with interpolation resolution $\epsilon = 5 \times 10^{-3}$ (refer to Figure 3) plotted against the embedding dimension D. Embedding lag is $\tau = 1600$. Error bars show the standard deviation of $\bar{\mu}_W$.

select an entropy threshold in the range $0.1259 \le \bar{\mu}_W \le 0.3855$ to discriminate between time series that are likely to be periodic from those that are likely to be chaotic (refer to the horizontal guide lines in Figure 7 corresponding to selected values of $\bar{\mu}_W$ for which λ_1 becomes positive). Finally, note the improvement in the $\bar{\mu}_W$ spectrum as a result of interpolation.

4. Conclusions

In this paper we have defined and discussed node aliasing, and how this undesirable effect can be minimised by using a sufficient degree of interpolation such that the network structure converges to a more accurate model of the dynamics. We have also proposed and investigated weighted average node entropy as a natural measure of dynamical complexity in time series, specifically with re-

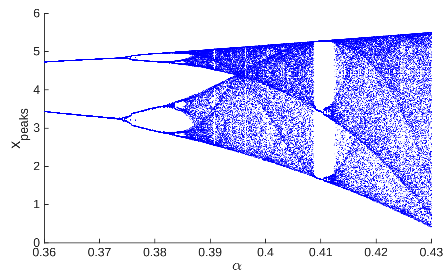

Figure 5: The bifurcation spectrum with respect to parameter α for the Rössler system.

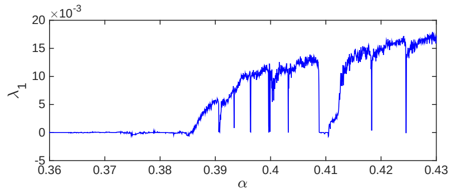

Figure 6: The largest Lyapunov exponent of the Rössler system plotted against the bifurcation parameter α, computed from the x-component time series using the lyap_k algorithm from the TISEAN software package [9].

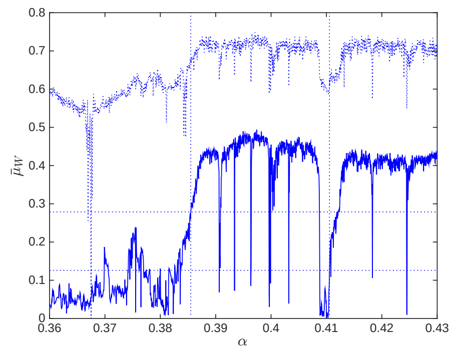

Figure 7: The weighted average node entropy $\bar{\mu}_W$ of ordinal partition networks from Rössler time series plotted against the bifurcation parameter α. The upper curve (dotted) is computed using the original time series that is sampled at intervals of 0.2 with embedding lag $\tau = 8$ and embedding dimension $D = 14$. The lower curve (solid) is computed using the same data but first interpolated using a cubic spline with interpolation resolution $\epsilon = 5 \times 10^{-3}$ (refer to Figure 3). The embdedding lag and dimension are constant at $\tau = 1600$ and $D = 14$. The two vertical lines denote the values of α at which the largest Lyapunov exponent λ_1 becomes positive after the period doubling cascade ($\alpha = 0.3855$) and the period-3 window ($\alpha = 0.4105$) respectively. Horizontal guides are drawn where these vertical lines cross the entropy curve at $(\alpha, \bar{\mu}_W) = (0.3855, 0.2778)$ and $(0.4105, 0.1259)$ as examples of possible entropy thresholds that could be used to demarcate likely ranges for periodic and chaotic dynamics in this data.

spect to transitional complexity over the partition of a reconstructed attractor, and demonstrated how this measure can track changes in dynamics across the bifurcation spectrum of the Rössler system.

Ongoing research efforts are currently focused on determining the robustness of the method to measurement noise, and the application of the method to experimental data from noise affected systems.

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