2009 International Symposium on Nonlinear Theory and its Applications
NOLTA'09, Sapporo, Japan, October 18-21, 2009

Numerical Uniqueness and Existence Theorem for Solution of Lippmann-Schwinger Equation

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Abstract—
In this paper, numerical uniqueness and existence theorem is presented for solution of Lippmann-Schwinger equation for sound scattering problem. A remarkable feature of this theorem is that the sufficient condition shown in this theorem can be evaluated by verified numerical computations.

1. Introduction

In this paper, we are concerned with the following scattering problem for the Helmholtz equation in the inhomogeneous media: find $u : \mathbb{R}^2 \to \mathbb{C}$ such that

$$\triangle u(x) + \kappa^2 b(x)u(x) = 0,\ x \in \mathbb{R}^2 \tag{1}$$

$$u = u^i + u^s, \tag{2}$$

$$\lim_{r=|x|\to\infty} r^{1/2}\left(\frac{\partial u^s}{\partial r} - \mathbf{i}\kappa u^s\right) = 0. \tag{3}$$

Here, $\mathbf{i} = \sqrt{-1}$ and κ is the wave number of the incidence wave. It is well known that this problem is given as a mathematical modeling of a two dimensional sound scattering problem [1].

We here assume the following:

Assumption 1 *The refractive index $b : \mathbb{R}^2 \to \mathbb{C}$ is a given smooth function and*

$$a(x) = b(x) - 1 \tag{4}$$

has a compact support such that

$$\textbf{supp}\ a \subset \overline{B}_\rho,\quad B_\rho = \{x \in \mathbb{R}^2 : |x| < \rho\}. \tag{5}$$

Furthermore, we assume that

$$a \in W^{\mu,2}(\mathbb{R}^2) \tag{6}$$

with $\mu > 1$.

Here, the Sobolev space $W^{\mu,2}(\mathbb{R}^2)$ consists of functions $w \in L^2(\mathbb{R}^2)$ such that

$$\int_{\mathbb{R}^2} (1 + |\xi|^2)^\mu |\hat{w}(\xi)|^2 d\xi < \infty, \tag{7}$$

where

$$\hat{w}(\xi) = \int_{\mathbb{R}^2} \mathbf{e}^{-\mathbf{i}x\cdot\xi} w(x)dx,\ \xi \in \mathbb{R}^2 \tag{8}$$

is the Fourier transform of w.

The incidence wave is assumed to be

$$u^i(x) = \exp(\mathbf{i}\kappa d \cdot x) \tag{9}$$

with a fixed $d \in \mathbb{R}^2$, $|d| = 1$. The incidence wave u^i is a plane wave solution of the Helmholtz equation

$$\triangle u(x) + \kappa^2 u(x) = 0. \tag{10}$$

The scattering wave u^s for this incidence wave is assumed to satisfy the Sommerfeld radiation condition (3).

It is known [1] that the problem defined by (1)-(3) is equivalent to the Lippmann-Schwinger equation

$$u(x) = \kappa^2 \int_{\mathbb{R}^2} E_\kappa(x - y)a(y)u(y)dy + u^i(x),\ x \in \mathbb{R}^2, \tag{11}$$

where $E_\kappa(x) = (\mathbf{i}/4)H_0^{(1)}(\kappa|x|)$. Here, $H_0^{(1)}$ is the Hankel function of the first kind of order zero. A fast solver for this equation has been proposed and studied mathematically in detail by Vainikko [2]. The monograph Saranen-Vainikko [3] has described Vainikko's theory in detail with its mathematical background. According to [2] and [3], first, we somewhat simplify and generalize (11). If we scale the independent variables x and y by $\tilde{x} = \kappa x$ and $\tilde{y} = \kappa y$, respectively, without loss of generality, we can assume that $\kappa = 1$. Further, instead of u^i, which is a solution of homogenized Helmholtz equation with $b(x) = 1$ for all $x \in \mathbb{R}^2$, we considier a general function f. We assume the following:

Assumption 2 $f \in W^{\mu,2}_{\mathbf{loc}}(\mathbb{R}^2), \mu > 1.$

Let us recall that $a \in W^{\mu,2}(\mathbb{R}^2)$ with $\textbf{supp}\ a \subset \overline{B}_\rho = \{x \in \mathbb{R}^2 : |x| \leqq \rho\}$. Then, the problem can be formulated as: find $u \in C(\overline{B}_\rho)$ satisfying the following equation

$$u(x) = \int_{B_\rho} E_1(x-y)a(y)u(y)dy + f(x),\ x \in B_\rho. \tag{12}$$

By the Sobolev embedding theorem, $W^{\mu,2}_{\mathbf{loc}} \subset C(\mathbb{R}^2)$ for $\mu > 1$. Multiplying both sides of (12) by $a(x)$, we can rewrite the equation with respect to $v(x) = a(x)u(x)$ as an unknown function:

$$v(x) = a(x)\int_{B_\rho} E_1(x-y)v(y)dy + a(x)f(x),\ x \in B_\rho. \tag{13}$$

A crucial observation is that, for $x \in \overline{B}_\rho$, only the values from $\overline{B}_{2\rho}$ of $E_1(x)$ are involved in the integral, *i.e.*, changing E_1 outside this ball, the solution $v(x)$ does not change in $\overline{B}_\rho$. We exploit this observation and redefine the kernel E_1 in $G_R\backslash\overline{B}_R$ where

$$G_R = \{x = (x_1, x_2) \in \mathbb{R}^2 : |x_1| < R, |x_2| < R\}$$

is an open box and $R > 2\rho$ is a parameter. Namely, we define

$$\mathcal{K}(x) = \begin{cases} E_1(x) = \dfrac{\mathbf{i}}{4}H_0^{(1)}(|x|), & |x| \leqq R \\ 0, & x \in G_R\backslash\overline{B}_R \end{cases} \tag{14}$$

After that we extend $\mathcal{K}, a, af$ and v from G_R to $\mathbb{R}^2$ as $2R$-periodic functions with respect to x_1 and x_2. Thus, we have arrived at the biperiodic integral equation

$$v(x) = a(x)\int_{G_R} \mathcal{K}(x-y)v(y)dy + a(x)f(x), \tag{15}$$

which is equivalent to (12).

The main theorem in [2] and [3] is the following:

Theorem 1 (Vainikko) *Assume that functions a and f satisfies Assumptions 1, 2 mentioned above and the homogeneous equation corresponding to (15) has in H^0 only the trivial solution. Then (15) has a unique solution $v \in H^\mu$ has a unique solution $v_N \in \mathcal{T}_N$ for sufficiently large N, and*

$$\|v_N - v\|_\lambda \leqq cN^{\lambda-\mu}\|v\|_\mu, \quad (0 \leqq \lambda \leqq \mu). \tag{16}$$

In [1], the existence of a unique solution in a function space of continuous functions is proved for the Lippmann-Schwinger equation in $\mathbb{R}^3$ using the unique continuation principle.

In this paper, for the sake of simplicity, we treat the case of $\mu \geqq 3/2$ and present Theorem giving a sufficient condition of guaranteeing that the homogeneous equation corresponding to (15) has in H^0 only the trivial solution. Thus, it can be seen as an another uniqueness theorem. A remarkable feature of this theorem is that the sufficient condition shown in this theorem can be evaluated by verified numerical computations. Furthermore, if such a sufficient condition holds, it also gives an upper bound of $\|(I-aK)^{-1}\|_{\mathcal{L}(H^0)}$ and a tight error bound between the exact solution v and an approximate solution $\tilde{v}$ which is generated by computer. As a result, this paper is to present a numerical uniqueness and existence theorem for (15), which asserts the existence of a unique solution around an approximate solution computed by numerical calculation.

2. Verification Theory

To present a numerical existence theorem of (15), we first prove several lemma.

Lemma 1 *Let $a \in H^\mu$ and $\mu \geqq \frac{3}{2}$. Then,*

$$\|aK\|_{\mathcal{L}(H^0, H^{\frac{3}{2}})} \leqq c_{\frac{3}{2}}c_R\|a\|_{\frac{3}{2}}. \tag{17}$$

Proof It is know that the following holds:

$$\|aKv\|_{\frac{3}{2}} \leqq c_{\frac{3}{2}}\|a\|_{\frac{3}{2}}\|Kv\|_{\frac{3}{2}} \tag{18}$$

Further, we have

$$\begin{aligned} \|Kv\|_{\frac{3}{2}} &= \left\|\int_{G_R} \mathcal{K}(x-y)v(y)dy\right\|_{\frac{3}{2}} \\ &= \sqrt{\sum_{j\in\mathbb{Z}^2} \underline{j}^3|\hat{F}(j)|^2} \\ &= \sqrt{\sum_{j\in\mathbb{Z}^2} \underline{j}^3|\hat{K}(j)|^2|\hat{v}(j)|^2} \\ &\leqq \sqrt{\sum_{j\in\mathbb{Z}^2} c_R^2|\hat{v}(j)|^2} \\ &= c_R\|v\|_0. \end{aligned} \tag{19}$$

Here, we have put

$$F(x) = \int_{G_R} \mathcal{K}(x-y)v(y)dy. \tag{20}$$

Further, we have used

$$|\mathcal{K}(j)| \leqq c_R\underline{j}^{-\frac{3}{2}}. \tag{21}$$

QED

Lemma 2 *Let $a \in H^\mu$ and $\mu \geqq \frac{3}{2}$. Then,*

$$\|Q_N(aK)\|_{\mathcal{L}(H^0,\mathcal{T}_N)} \leqq c_{\frac{3}{2}}c_R\|a\|_{\frac{3}{2}}. \tag{22}$$

Proof

$$\begin{aligned} \|Q_N(aKv)\|_0 &\leqq \|aKv\|_{\frac{3}{2}} \\ &= c_{\frac{3}{2}}c_R\|a\|_{\frac{3}{2}}\|v\|_0 \end{aligned} \tag{23}$$

QED

Theorem 2 *Let $a \in H^\mu$ and $\mu \geqq \frac{3}{2}$. Let $k = c_{\frac{3}{2}}c_R\|a\|_{\frac{3}{2}}$ and $c_N = c_{0,\frac{3}{2}}N^{-\frac{3}{2}}$. If*

$$\|(I - Q_N aK)^{-1}\|_{\mathcal{L}(\mathcal{T}_N)} \leqq M \tag{24}$$

and

$$c_N k(1 + Mk) < 1, \tag{25}$$

the operator $I - aK$ is invertible and the following is satisfied:

$$\|(I - aK)^{-1}\|_{\mathcal{L}(H^0)} \leqq C_M. \tag{26}$$

Here,

$$C_M = \frac{1 + Mk}{1 - c_N k(1 + Mk)}. \tag{27}$$

From Therorem 2, it follows the following theorem:

Theorem 3 *Let* $a \in H^{\mu}$ *and* $\mu \geqq \frac{3}{2}$. *Let* $\tilde{v} \in \mathcal{T}_N$ *be an approximate solution of*

$$(I - aK)v = af. \tag{28}$$

If

$$\|(I - Q_N aK)^{-1}\|_{\mathcal{L}(\mathcal{T}_N)} \leqq M \tag{29}$$

and

$$c_N k(1 + Mk) < 1 \tag{30}$$

hold, then (28) has a unique solution v^* *satisfying*

$$\|\tilde{v} - v^*\|_{H^0} \leqq C_M \|(I - aK)\tilde{v} - af\|_{H^0}. \tag{31}$$

Furthermore, $v^* \in H^{\frac{3}{2}}$.

References

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