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Computer assisted proofs of solutions to Nonlinear elliptic partial differential equations

Akitoshi Takayasu†, Shin'ichi Oishi‡ and Takayuki Kubo§

†Graduate School of Fundamental Science and Engineering, Waseda University
3-4-1 Okubo, Shinjuku-ku, Tokyo 169-8555, Japan
‡Faculty of Science and Engineering, Waseda University & CREST, JST,
3-4-1 Okubo, Shinjuku-ku, Tokyo 169-8555, Japan
§Institute of Mathematics, University of Tsukuba,
Tenodai 1-1-1, Tsukuba, Ibaraki 305-8571 Japan
Email: takitoshi@suou.waseda.jp, oishi@waseda.jp, tkubo@math.tsukuba.ac.jp

Abstract—In this article, a numerical method is presented for computer assisted proofs to the existence and uniqueness of solutions to Dirichlet boundary value problems in a certain class of nonlinear elliptic equations. In a weak formulation of the problem, a weak solution is described as a zero point of a certain nonlinear map. Based on Newton-Kantorovich theorem, a numerical existence and local uniqueness of solutions are proved by our proposed method. Some conditions need to be checked numerically. It is shown that all errors of numerical computations such as discretization errors and rounding errors are figured out by numerical computations with result verification. Finally, an illustrative numerical result is presented for showing the usefulness of proposed method.

1. Introduction

Let Ω be a bounded convex polygonal domain in $\mathbb{R}^m$ with $m = 2, 3$. This article is concerned with Dirichlet boundary value problems of nonlinear elliptic equations:

$$\begin{cases} -\nabla \cdot (a\nabla u) = f(u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1}$$

where $a(x)$ is a smooth function on $\overline{\Omega}$ with $a(x) \geqq a_0 > 0$ for some $a_0 \in \mathbb{R}$. Here, $f : H_0^1(\Omega) \to L^2(\Omega)$ is assumed to be Fréchet differentiable. For example, the following function

$$f(u) = -b \cdot \nabla u - cu + c_2 u^2 + c_3 u^3 + g$$

with $b(x) \in (L^\infty(\Omega))^m$, $c, c_2, c_3 \in L^\infty(\Omega)$ and $g \in L^2(\Omega)$ satisfies this condition. We shall propose a numerical method of computer assisted proofs for the existence and local uniqueness of solutions to the problem (1).

In 1988, M. T. Nakao [1] has presented a method of a computer assisted proof for elliptic problems. In 1991, Plum [2] has also presented another method of proving the existence and uniqueness of solutions for the problem (1). Both methods have been demonstrated to be useful in this two decades. On the other hand, this article presents another method of computer assisted proofs for (1) based on the finite element method. In the following section, we describe how to work the proposal method. Then a computer assisted proof algorithm is presented in Section 3. Finally, we can show the illustrative result of our method.

2. Outline of proposal computer assisted proofs

In this part, we shall briefly sketch our proposed numerical method to prove the existence of weak solutions for (1). Proposed method also evaluates guaranteed error bounds in which there is an unique solution of original equations.

Let $H^{-1}(\Omega)$ be the topological dual space of $H_0^1(\Omega)$ the space of linear continuous functionals on $H_0^1(\Omega)$. For $u, v \in H_0^1(\Omega)$, let us define a continuous bilinear form $A(u, v)$ as

$$A(u, v) = (a\nabla u, \nabla v).$$

If we fix $u \in H_0^1(\Omega)$, then $A(u, \cdot) \in H^{-1}(\Omega)$. Thus, we can define an operator $\mathcal{A} : H_0^1(\Omega) \to H^{-1}(\Omega)$ by

$$< \mathcal{A}u, v >= A(u, v).$$

Let us define

$$\|u\|_a = \sqrt{A(u, u)}.$$

This norm is equivalent to H_0^1-norm, *i.e.*, there exist positive constants C_a and c_a satisfying

$$c_a \|u\|_{H_0^1} \leqq \|u\|_a \leqq C_a \|u\|_{H_0^1} \quad \text{for} \quad u \in H_0^1(\Omega),$$

In fact, we can choose $c_a = \sqrt{a_0}$ and $C_a = \sqrt{\|a\|_{L^\infty}}$. Further, we can define an operator $\mathcal{N} : H_0^1(\Omega) \to H^{-1}(\Omega)$ by

$$< \mathcal{N}u, v >= N(u, v) = (f(u), v),$$

Then, a weak form of Eq. (1) can be written as

$$\mathcal{A}u = \mathcal{N}u.$$

Now, let us define the operator $\mathcal{F} : H_0^1(\Omega) \to H^{-1}(\Omega)$ by

$$\mathcal{F}u = (\mathcal{A} - \mathcal{N})u.$$

Then, Eq.(2) can be written as

$$\mathcal{F}u = 0. \tag{2}$$

In the following, we will discuss how to prove the existence and uniqueness of the solution of Eq. (2), the weak solution of the problem (1). Newton-Kantorovich theorem is applicable to the nonlinear operator equation (2). This theorem gives our desired computer assisted proof for the existence and local uniqueness of solutions to Eq. (2).

In order to apply Newton-Kantorovich theorem, the Fréchet derivative of $\mathcal{F}$ is needed. The Fréchet differentiability of $\mathcal{F}$ is followed by that of f. Moreover, the Fréchet derivative of $\mathcal{N}$ at $\hat{u} \in H_0^1(\Omega)$, $\mathcal{N}'(\hat{u}) : H_0^1(\Omega) \to H^{-1}(\Omega)$, is given by

$$< \mathcal{N}'(\hat{u})u, v >= N'(\hat{u})(u, v) = (f'(\hat{u})u, v).$$

Here, $f'(\hat{u}) : H_0^1(\Omega) \to L^2(\Omega)$ is the Fréchet derivative of $f : H_0^1(\Omega) \to L^2(\Omega)$ at $\hat{u}$. Thus, for a given $u \in H_0^1(\Omega)$ the Fréchet derivative $\mathcal{F}'(u) : H_0^1(\Omega) \to H^{-1}(\Omega)$ is given as

$$\mathcal{F}'(u) = \mathcal{A} - \mathcal{N}'(u).$$

Now, we assume that an approximate solution $\hat{u} \in H_0^1(\Omega)$ is given. The existence and uniqueness of solution are proved by computer assisted proof in the neighborhood of $\hat{u}$. The following Newton-Kantorovich theorem is applicable. This form of Newton-Kantorovich theorem is called an affine invariant form [3].

Theorem 1 (Newton-Kantorovich Theorem) *Let* $\hat{u} \in H_0^1(\Omega)$. *Let* $\mathcal{F} : H_0^1(\Omega) \to H^{-1}(\Omega)$ *be Fréchet differentiable at* $\hat{u}$. *Assume that the Fréchet derivative* $\mathcal{F}'(\hat{u})$ *is nonsingular and satisfies*

$$\|\mathcal{F}'(\hat{u})^{-1}\mathcal{F}\hat{u}\|_{H_0^1} \leqq \alpha,$$

for a certain positive α. *Then, let* $\mathcal{F} : H_0^1(\Omega) \to H^{-1}(\Omega)$ *be Fréchet differentiable on* $B(\hat{u}, 2\alpha) = \{v \in H_0^1(\Omega) : \|v - \hat{u}\|_{H_0^1} \leqq 2\alpha\} \subset H_0^1(\Omega)$ *and assume that for a certain positive* ω *and for any* $v, w \in B(\hat{u}, 2\alpha)$, *the following holds:*

$$\|\mathcal{F}'(\hat{u})^{-1}(\mathcal{F}'(v) - \mathcal{F}'(w))\|_{\mathcal{L}(H_0^1,H_0^1)} \leqq \omega\|v - w\|_{H_0^1}.$$

If

$$\alpha\omega \leqq \frac{1}{2},$$

then there is a solution $u^* \in H_0^1(\Omega)$ *of* $\mathcal{F}u = 0$ *satisfying*

$$\|u^* - \hat{u}\|_{H_0^1} \leqq \rho := \frac{1 - \sqrt{1 - 2\alpha\omega}}{\omega}.$$

Furthermore, the solution u^* *is unique in* $B(\hat{u}, \rho)$. □

Next let us define the finite element approximation and discrete projection with respect to the mesh size h. Let X_n denote a finite-dimensional space spanned by linearly independent H_0^1-conforming finite element base functions $S_h = \{\phi_1, \phi_2, \cdots, \phi_n\}$ depending on the mesh size h, $(0 < h < 1)$:

$$X_n = \text{span}\{\phi_1, \phi_2, ..., \phi_n\} \subset H_0^1(\Omega).$$

For example, the dimension of X_n is $(k + 1)^2$ if we take an uniform mesh on square domain with k-equidistant partition by piecewise linear base functions.

The Ritz-projection $\mathcal{P}_n : H_0^1(\Omega) \to X_n$ is defined by

$$\Big(a(x)(\nabla u - \nabla(\mathcal{P}_n u)), \nabla\phi_h\Big) = 0, \quad \forall\, \phi_h \in X_n. \tag{3}$$

For $u \in H_0^1(\Omega) \cap H^2(\Omega)$ and its approximation $\mathcal{P}_n u \in X_n$, the error estimate is given as

$$\|u - \mathcal{P}_n u\|_{H_0^1} \leqq C_0(h)\|f(u)\|_{L^2}.$$

In case of $a(x) = 1$, for the rectangular mesh, Nakao, Yamamoto and Kimura [4] have shown that one can take $C_0(h) = \frac{h}{\pi}$ for bilinear element. Kikuchi and Liu [5] have proved that for $a(x) = 1$ and for the linear and equilateral triangle mesh of the convex polygonal domain, $C_0(h)$ can be taken as $0.493h$.

Now, we consider a finite dimensional approximation of Eq. (2) of the following form:

$$\mathcal{P}_n\mathcal{A}^{-1}\mathcal{F}\mathcal{P}_n u = \mathcal{P}_n\mathcal{A}^{-1}(\mathcal{A}-\mathcal{N})\mathcal{P}_n u = \mathcal{P}_n(u-\mathcal{A}^{-1}\mathcal{N}\mathcal{P}_n u) = 0.$$

Let $u_h \in X_n$ be a solution of

$$\mathcal{P}_n(u_h - \mathcal{A}^{-1}\mathcal{N}\mathcal{P}_n u_h) = 0. \tag{4}$$

Eq. (4) becomes

$$(a(x)\nabla u_h, \nabla\phi_h) = (f(u_h), \phi_h), \quad (\forall\phi_h \in X_n),$$

which is nothing but the finite element approximation [6] of the nonlinear equation (2).

We also discuss how to calculate constants α and ω in Theorem 1. Three constants are needed to evaluate. One is the inverse operator norm estimation.

$$\|\mathcal{F}'(\hat{u})^{-1}\|_{\mathcal{L}(H^{-1},H_0^1)} = \|(\mathcal{A} - \mathcal{N}'(\hat{u}))^{-1}\|_{\mathcal{L}(H^{-1},H_0^1)} \leq C_1,$$

This is estimated by the following theorem given by S. Oishi [7]. This theorem is based on perturbation lemma of linear operators [8] and given as

Theorem 2 (Oishi 1995) *Let* $\hat{u} \in H_0^1(\Omega)$ *and* $\mathcal{N}'(\hat{u}) : H_0^1(\Omega) \to H^{-1}(\Omega)$ *be a linear compact operator. Let* X_n *be a finite dimensional subspace of* $H_0^1(\Omega)$ *spanned by the finite element bases* $S_h = \{\phi_1, \phi_2, \ldots, \phi_n\}$. *Let* $\mathcal{P}_n : H_0^1(\Omega) \to X_n$ *be the Ritz-projection. Assuming that* $\mathcal{P}_n\mathcal{A}^{-1}\mathcal{N}'(\hat{u}) : H_0^1(\Omega) \to H_0^1(\Omega)$ *is bounded and satisfies*

$$\|\mathcal{P}_n\mathcal{A}^{-1}\mathcal{N}'(\hat{u})\|_{\mathcal{L}(H_0^1,H_0^1)} \leqq K,$$

the difference between $\mathcal{A}^{-1}\mathcal{N}'(\hat{u})$ *and* $\mathcal{P}_n\mathcal{A}^{-1}\mathcal{N}'(\hat{u})$ *is bounded and enjoys*

$$\|(\mathcal{A}^{-1} - \mathcal{P}_n\mathcal{A}^{-1})\mathcal{N}'(\hat{u})\|_{\mathcal{L}(H_0^1,H_0^1)} \leqq L$$

and the finite dimensional operator $\mathcal{P}_n(\mathcal{I} - \mathcal{A}^{-1}\mathcal{N}'(\hat{u}))|_{X_n} : X_n \to X_n$ is invertible with

$$\|(\mathcal{P}_n(\mathcal{I} - \mathcal{A}^{-1}\mathcal{N}'(\hat{u}))|_{X_n})^{-1}\|_{\mathcal{L}(H_0^1,H_0^1)} \leqq M.$$

Here, $\mathcal{P}_n(\mathcal{I} - \mathcal{A}^{-1}\mathcal{N}'(\hat{u}))|_{X_n} : X_n \to X_n$ is the restriction of the operator $\mathcal{P}_n(\mathcal{I} - \mathcal{A}^{-1}\mathcal{N}'(\hat{u})) : H_0^1(\Omega) \to X_n$ on X_n. If $(1 + MK)L < 1$, then the operator $\mathcal{A} - \mathcal{N}'(\hat{u}) : H_0^1(\Omega) \to H^{-1}(\Omega)$ is also invertible and

$$\|(\mathcal{A} - \mathcal{N}'(\hat{u}))^{-1}\|_{\mathcal{L}(H^{-1},H_0^1)} \leqq \frac{1}{c_a^2}\frac{1 + MK}{1 - (1 + MK)L} =: C_1.$$

□

For computer assisted proofs, next constant is the residual estimation of the operator equation (2). It is bounded by

$$\begin{aligned}\|\mathcal{F}\hat{u}\|_{H^{-1}} &= \|\mathcal{A}\hat{u} - \mathcal{N}\hat{u}\|_{H^{-1}} \\ &\leqq \|\hat{u} - \mathcal{P}_n\mathcal{A}^{-1}\mathcal{N}(\hat{u})\|_{H_0^1} + C_0(h)\|f(\hat{u})\|_{L^2} =: C_2.\end{aligned}$$

Further, the Lipschitz constant is the last constant to evaluate. It is defined through

$$\|\mathcal{F}'(v) - \mathcal{F}'(w)\|_{\mathcal{L}(H^{-1},H_0^1)} \leqq C_3\|v - w\|_{H_0^1}.$$

Then it follows that $\alpha \leqq C_1C_2$ and $\omega \leqq C_1C_3$. If $\alpha\omega \leqq C_1^2C_2C_3 < \frac{1}{2}$ is obtained by verified computation, then the existence and uniqueness of the solution are proved numerically. So that Newton-Kantorovich theorem can be applied to the nonlinear operator equation (2) if three constants are estimated.

3. Computer assisted existence test

In this section, summing up the above discussions, an algorithm of computer assisted proofs is described for verifying the existence and local uniqueness of solutions to Eq. (1) in the neighborhood of $\hat{u}$.

Algorithm 1 (Nonlinear Elliptic Eq.)
1. Compute an approximate solution $\hat{u} \in H_0^1(\Omega)$ of the problem (4)
2. Compute rigorous upper bound of $\|(\mathcal{A} - \mathcal{N}'(\hat{u}))^{-1}\|_{\mathcal{L}(H^{-1},H_0^1)}$ by the following steps:

2.1 Compute $\|\hat{u}\|_{L^\infty}$ and calculate K, L by

$$K = \frac{C_{e,2}}{a_0}\|f'(\hat{u})\|_{\mathcal{L}(H_0^1,L^2)},$$

and

$$L = C_0(h)\|f'(\hat{u})\|_{\mathcal{L}(H_0^1,L^2)},$$

respectively [1].

2.2 Let D and G be $n \times n$ matrices whose i-j elements are given by

$$(a(x)\nabla\phi_j, \nabla\phi_i),$$

and

$$(a(x)\nabla\phi_j, \nabla\phi_i) - (f'(\hat{u})\phi_j, \phi_i),$$

respectively. Let a lower triangular matrix $\hat{L}$ be the Cholesky decomposition of D, $D = \hat{L}\hat{L}^t$. If G is invertible, then set

$$M = \frac{C_a}{c_a}\|\hat{L}^tG^{-1}\hat{L}\|_2.$$

When G is not invertible, stop with failure.

2.3 Check whether $(1 + MK)L < 1$ holds or not. If this holds, then by Theorem 2

$$\|(\mathcal{A}-\mathcal{N}'(\hat{u}))^{-1}\|_{\mathcal{L}(H^{-1},H_0^1)} \leqq \frac{1 + MK}{a_0(1 - (1 + MK)L)} =: C_1.$$

Otherwise, stop with failure.

3. Calculate the residual by the formula

$$C_2 := C_a^2\left(\|\hat{u} - \mathcal{P}_n\mathcal{A}^{-1}\mathcal{N}(\hat{u})\|_{H_0^1} + C_0(h)\|f(\hat{u})\|_{L^2}\right).$$

Set $\alpha = C_1C_2$.
4. Calculate the Lipschitz constant C_3 by

$$C_3 := \left(\frac{C_a}{c_a}\right)^2 C_{e,2}C_L$$

where C_L is the Lipschitz constant of f'.
Set $\omega = C_1C_3$.
5. Check the condition $\alpha\omega \leqq \frac{1}{2}$. If this condition is satisfied, there is a solution $u^* \in H_0^1(\Omega)$ of $\mathcal{F}u = 0$ satisfying

$$\|u^* - \hat{u}\|_{H_0^1} \leqq \rho := \frac{1 - \sqrt{1 - 2\alpha\omega}}{\omega}.$$

Further, the solution u^* is unique in $B(\hat{u}, \rho)$.
Otherwise, stop with failure.

4. Computational result

Now, we shall present a numerical result to illustrate the usefulness of our method. All computations are carried out on Mac OS X, 2.26GHz Quad-Core Intel Xeon by using MATLAB 2010a with a toolbox for verified computations, INTLAB [9].

For an application of our computer assisted proof method, we treat a nonlinear Dirichet boundary value problem on $\Omega = (0, 1) \times (0, 1)$:

$$\begin{cases} -\Delta u = u^3 + 5 & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega. \end{cases} \tag{5}$$

Obviously, the Fréchet derivative of $f(u) = u^3 + 5 : H_0^1(\Omega) \to L^2(\Omega)$ is given by $f'(u) = 3u^2$. An approximate solution $\hat{u}$ is calculated by the finite element method

[1] $C_{e,p}$ denotes Poincaré constant $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$ by Sobolev embedding theorem, *ex.* $\|u\|_{L^2} \leqq C_{e,2}\|u\|_{H_0^1}$ for $u \in H_0^1(\Omega)$.

(pde toolbox on MATLAB) with uniform piecewise linear elements of the mesh size $\frac{1}{32}$. The approximate solution is bounded on Ω then $\hat{u}$ is the element of $L^\infty(\Omega)$ in this solution. So that for $\hat{u} \in L^\infty(\Omega) \cap H_0^1(\Omega)$, we can use

$$\|f'(\hat{u})\|_{\mathcal{L}(H_0^1,L^2)} \leqq 3 \min \left\{ C_{e,2}\|\hat{u}\|_{L^\infty}^2, C_{e,6}^3\|\hat{u}\|_{H_0^1}^2 \right\} \|c_3\|_{L^\infty}.$$

For the Lipschitz continuity of $\mathcal{F}'(u)$, for $u \in H_0^1(\Omega)$ and $v, w \in B(\hat{u}, 2\alpha)$ we have

$$\begin{aligned}\|(f'(v) - f'(w))u\|_{L^2} &\leqq \|3c_3(v+w)(v-w)u\|_{L^2} \\ &\leqq \left(3C_{e,6}^3\|c_3\|_{L^\infty}\|v+w\|_{H_0^1}\right)\|v-w\|_{H_0^1}\|u\|_{H_0^1}.\end{aligned}$$

Since $v, w \in B(\hat{u}, 2\alpha)$, it follows that

$$\|v+w\|_{H_0^1} \leqq 2\|\hat{u}\|_{H_0^1} + 4\alpha.$$

Thus, we have

$$\|f'(v) - f'(w)\|_{\mathcal{L}(H_0^1,L^2)} \leqq C_L\|v-w\|_{H_0^1(\Omega)},$$

with

$$C_L = 6C_{e,6}^3\|c_3\|_{L^\infty}\left(\|\hat{u}\|_{H_0^1} + 2\alpha\right).$$

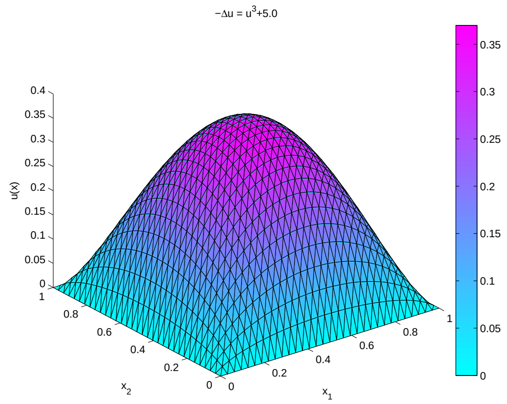

Fig.1: Apptroximate solution $\hat{u}$ ($h = \frac{1}{32}$)

The verification algorithm in the previous section is applicable to Eq. (5) with mesh size $\frac{1}{32}$. Our computer assisted proof yields

$$\|u - \hat{u}\|_{H_0^1} \leqq \rho = 8.85 \times 10^{-2}.$$

Consequently, it follows that there exists an unique solution in the ball $B(\hat{u}, \rho)$

By increasing grid points, guaranteed error bounds are improved with the ratio almost $O(h)$. The guaranteed error is presented in Table1.

Table 1: Computational results to Problem (5)

Mesh size: $\frac{1}{2^x}$	Error: E_x	$O(h^\gamma)$
4	2.07×10^{-1}	-
5	8.85×10^{-2}	1.17
6	4.22×10^{-2}	1.05
7	2.07×10^{-2}	1.02
8	1.03×10^{-2}	1.01
9	5.11×10^{-3}	1.00

Furthermore we focus on the computational cost of proposal method. It is several (up to 10) times more than that of the approximation. An illustrative result with respect to the performance is presented in Table 2. Here, we assume that t_1 is a computing time to get approximate solution.

Table 2: Comparing the execution time

Mesh size: $\frac{1}{2^x}$	Approximate (t_1)	Verification ($/t_1$)
4	0.04	2.09
5	0.11	1.99
6	0.47	2.21
7	2.14	3.19
8	17.25	3.96
9	89.21	8.64

Table 2 states that only some additional costs (up to 10 times) cause the verified solution by our computer assisted proof method.

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