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Analysis of Phase-Inversion Waves in Coupled Piecewise Constant Oscillators as a Ladder

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Abstract—In this study, we consider phase-inversion waves in coupled piecewise-constant oscillators as a ladder. The phase-inversion waves are a kind of wave propagation phenomena, and the phase states of the waves are propagated to next oscillator in succession. In this paper, we show that phase-inversion waves an be observed in coupled three piecewise-constant oscillators by using the experimental circuit and rigorous solutions . We also analyze the stability of the phase-inversion waves in our system by Lyapunov exponent obtained by a computational algorithm for piecewise-constant systems.

1. Introduction

There are many reports for analysis of synchronization phenomena of coupled oscillators[1]-[3]. Suzuki et al. have confirmed that piecewise-constant oscillators coupled by hysteresis elements exhibit co-existence of in-phase and anti-phase synchronization[1]. They also analyzed the stability of the system by Lyapunov exponents. Yamauchi et al. have discovered wave propagation phenomena called phase-inversion waves of coupled van der Pol oscillators[2][3]. The phase-inversion waves are a kind of phase-wave, and the phase states of the waves are propagated to next oscillator in succession. It is very important to analyze the these phenomena, because it is similar to propagation phenomena of electrical information in an axial fiber of nervous systems. However, if nonlinearity of van der Pol oscillators are strong, the analysis often becomes hard. Therefore, confirmation of phase-inversion waves in rigorous sense and detailed stability analysis of the systems which the phase-inversion waves are generated have not been discussed. Accordingly, we consider piecewise-constant oscillators. The oscillators are simple systems and the analysis is relatively easy. Piecewise-constant systems have piecewise-constant vector fields, and the solutions are piecewise-linear. Hence, we have only to focus on the borders of switching of the vector fields, we can determine the rigorous solutions[1]. In this paper, we show phase-inversion waves of coupled three piecewise-constant oscillators. We also analyze the stability of the phase-inversion waves in our system by Lyapunov exponents. They can be obtained rigorously from computer-aided analyzing procedure by using rigorous solutions. Typical result are confirmed in laboratory.

2. Circuit model

2.1. A Piecewise-Constant Oscillator

Figure 1 shows a circuit model of a piecewise-constant oscillator.

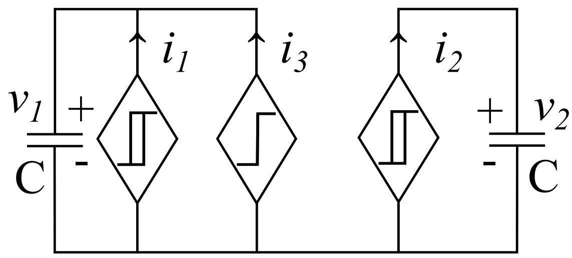
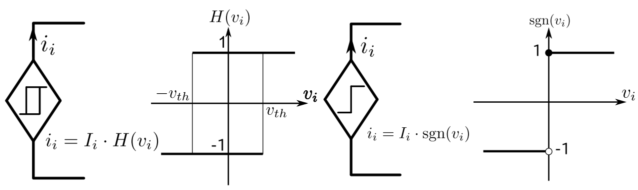


Fig. 1 Cicuit model of a piecewise-constant oscillator.



(a)VCCS with hysteresis characteristic.

(b)VCCS with Signum-like function.

Fig. 2 Symbols and nonlinear characteristics of VCCSs.

The circuit equations of the system are described as follows.

$$\begin{cases} C\dfrac{\mathrm{d}v_1}{\mathrm{d}t} = I_1 \cdot H(v_1) + I_3 \cdot \mathrm{sgn}(v_2), \\ C\dfrac{\mathrm{d}v_2}{\mathrm{d}t} = I_2 \cdot H(v_1), \end{cases} \quad (1)$$

where I_1, I_2 and I_3 are absolute values of output currents of hysteresis or sgn Voltage Controlled Current

Sources(VCCSs). We consider the following conditions.

$$I_2 = -I_3,\ \ I_1 \cdot I_2 < 0. \tag{2}$$

The conditions (2) guarantees non-constrained behaviors. $H(v_{in})$ and $\mathrm{sgn}(v_{in})$ are hysteresis and signum characteristic respectively, as shown in Fig. 2. We use following dimensionless variables and parameters

$$\tau = \frac{I_2}{C \cdot v_{th}}t,\ x = \frac{v_1}{v_{th}},\ y = \frac{v_2}{v_{th}},\ \alpha = -\frac{I_1}{I_2}. \tag{3}$$

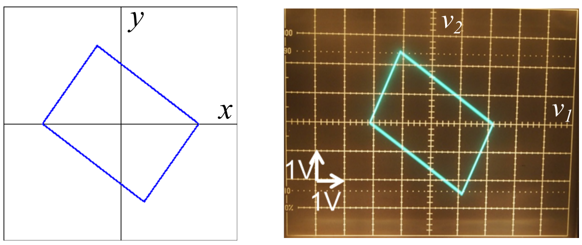
Then, we can rewrite the circuit eq. (1) as following dynamics,

$$\begin{cases} \dot{x} = -\alpha h(x) - \mathrm{sgn}(y) \\ \dot{y} = h(x), \end{cases} \tag{4}$$

where "·" denote differentiation by normalized time τ, $h(X)$ shows normalized hysteresis. If X reaches 1, the output switches from -1 to 1, and if X reaches -1, output switches from 1 to -1. The system has only one parameter α. In order to oscillate, we consider the following conditions[1].

$$0 < \alpha < 1. \tag{5}$$

Figure 3 shows a rigorous solution and the corresponding laboratory measurement.



(a)Rigorous solution. (b)Laboratory measurement.

Fig. 3 Rigorous solution and laboratory measurement of a piecewise-constant oscillator.

2.2. Piecewise Constant Oscillators Coupled by hysteresis element as a Ladder

We consider piecewise-constant oscillators coupled by hysteresis elements as a ladder. Figure 4 shows circuit model of the coupled piecewise-constant oscillators. This

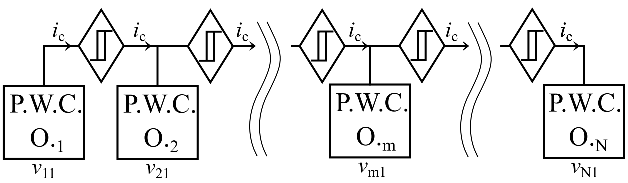


Fig. 4 Circuit model of coupled piecewise-constant oscillators by hysteresis elements.

system has following dynamics.

[First oscillator] $(m = 1)$

$$\begin{cases} \dot{x_1} = -\alpha_1 h(x_1) - \mathrm{sgn}(y_1) - \gamma h(x_1 - x_2) \\ \dot{y_1} = h(x_1), \end{cases} \tag{6}$$

[Middle oscillator] $(2 \le m \le N-1)$

$$\begin{cases} \dot{x_m} = -\alpha_m h(x_m) - \mathrm{sgn}(y_m) \\ \qquad -\gamma h(x_m - x_{m+1}) + \gamma h(x_{m-1} - x_m) \\ \dot{y_m} = h(x_m), \end{cases} \tag{7}$$

[Last oscillator] $(m = N)$

$$\begin{cases} \dot{x_N} = -\alpha_N h(x_N) - \mathrm{sgn}(y_N) \\ \qquad +\gamma h(x_{N-1} - x_N) \\ \dot{y_N} = h(x_N). \end{cases} \tag{8}$$

The system has $N+1$ parameters, $\alpha_m (1 \le m \le N)$ and γ. N is number of oscillators, and γ is a coupling parameter. In this paper, we discuss the case of $N = 3$.

3. Algorithm for the rigorous solution

Since the system is a piecewise-constant system, we can obtain the rigorous solution directly by using mapping procedure from border to border[1]. For easy to explain, we introduce the algorithm for the rigorous solution be given a piecewise-constant oscillator.

STEP1 .

Let initial value $\mathbf{x_0}$ be substituted into eq. (9).

$$i = \frac{-h(x)+1}{2} \cdot 2^1 + \frac{-\mathrm{sgn}(y)+1}{2} \cdot 2^0,\ \ i \in 0, 1, 2, 3, \tag{9}$$

where, $\mathbf{x_0} = {}^t(x, y)$, and ^{t} denote the transpose of vector. In the case of the piecewise-constant oscillator, the dynamics is controlled by four vector fields represented Table 1. i is the dependent variable to indicate each vector fields.

STEP2 .

We calculate a time τ_k until $\mathbf{x_0}$ reaches border. Assuming $\mathbf{x_0}$ hits border of $E_x(i_k)$ or $E_y(i_k)$ and each reach time τ_x, τ_y are obtained by following equations, because trajectory is the manner of linear uniform motion.

$$\tau_x = \frac{E_x(i_k) - x}{\dot{x}},\ \tau_y = \frac{E_y(i_k) - y}{\dot{y}}, \tag{10}$$

where $E_x(i_k)$, $E_y(i_k)$ are border of vector fields. In these number, positive minimum value is the time until next border τ_k.

STEP3 .

We calculate $\mathbf{x_{k+1}}$. It is obtained by

$$\mathbf{x_{k+1}} = \mathbf{x_k} + \boldsymbol{a}(i_k) \cdot \tau_k. \tag{11}$$

where $\boldsymbol{a}(i)$ is vector fields corresponding i. Therefore, $\dot{\mathbf{x_k}} = \boldsymbol{a}(i)$. After switching, We calculate i_{k+1}.

STEP4 .

Let $\boldsymbol{x_{k+1}}$ and i_{k+1} be replaced with $\boldsymbol{x_k}$, i_k, return to STEP2.

Table 1: Local vector fields and borders for i.

i	$h(x)$	sgn(y)	$\boldsymbol{a}(i)$	$E_x(i)$	$E_y(i)$
0	1	1	${}^t(-\alpha-1, 1)$	-1	0
1	1	-1	${}^t(-\alpha+1, 1)$	-1	0
2	-1	1	${}^t(\alpha-1, -1)$	1	0
3	-1	-1	${}^t(\alpha+1, -1)$	1	0

The system repeats the manner of linear uniform motion and switching. All trajectories started from any initial states must converge to a square limit cycle as shown Fig. 3. It is also possible to derive the exact solution by using the same algorithm even coupled systems.

4. Stability Analysis by Lyapunov exponent

In this section, we use the largest Lapunov exponent for stability analysis of our system. Lapunov exponents can be obtained by Jacobian matrix that determined from vector fields $\boldsymbol{a}(i)$. Jacobian matrix is determined by transformed from eq.(11) to following:

$$\boldsymbol{x_{k+1}} = A \cdot \boldsymbol{x_k}, \tag{12}$$

where A corresponds to Jacobian matrix J_k. The largest Lapunov exponent is given by

$$\begin{aligned} \lambda_1 &= \lim_{L\to 4500000} \frac{1}{L}\sum_{k=0}^{L} \ln\|J_k e_k\|, \\ e_{k+1} &= \frac{J_k e_k}{\|J_k e_k\|}, \end{aligned} \tag{13}$$

where e_k is orthonormal base[4], and L is iteration number. We set L to 4500000 as large enough. Figure 5 represents the largest Lyapunov exponents associated with changing α_2, where α_2 is a parameter of the middle oscillator. The upper row of Fig. 5 shows a result by increasing α_2. Similarly, lower is by decaying α_2, where $\alpha_1 = \alpha_3 = 0.33$, $\gamma = -0.02$. In near region of $\alpha_2 = 0.354$, the largest Lyapunov exponents are true for both positive and negative values. Figure 6 represents the difference between output voltages of adjacent oscillators in these parameter ranges. As shown in Fig. 6, we can confirm co-existence phenomenon of phase-inversion wave and anti-phase synchronization without changing phase states. We calculate the largest Lyapunov exponents when co-existence phenomenon happens. As a result, the largest Lyapunov exponent is $\lambda_1 = 0.003378$ in Fig. 6(a) if phase-inversion wave happens, that is the system is unstable. On the other hand, the largest Lyapunov exponent is $\lambda_1 = -0.008793$ in Fig. 6(b) if anti-phase synchronization happens, namely, the system is stable. In other parameters, if $\lambda_1 < 0$, phase-inversion waves do not happen. Figure 7 represent the dif-

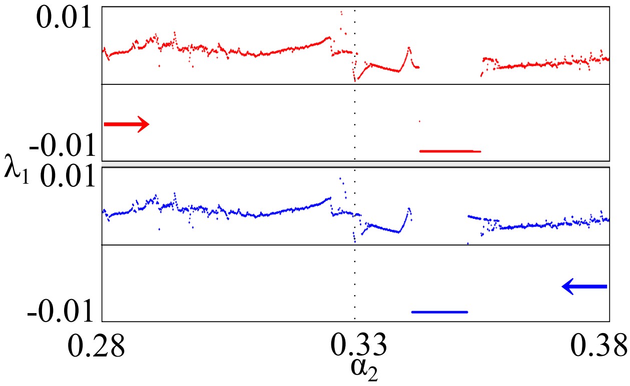


Fig. 5 The largest Lyapunov exponents to changes in α_2

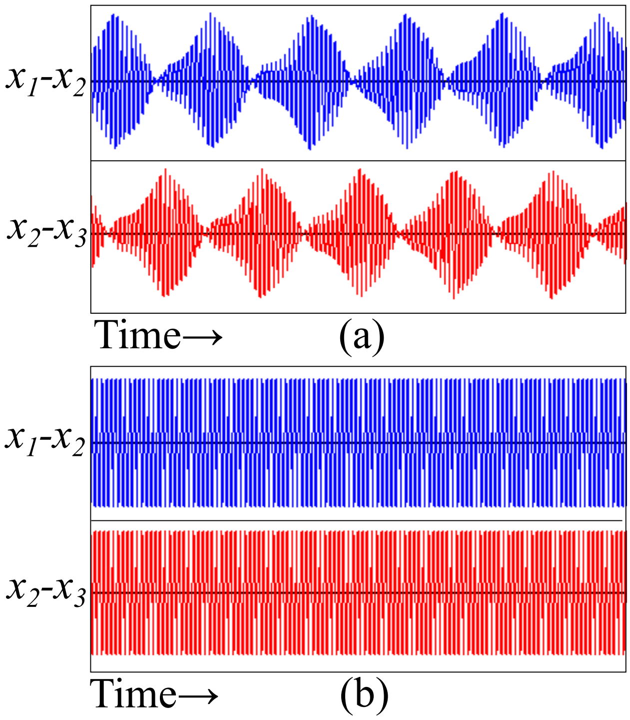


Fig .6 The difference between output of adjacent oscillators in $\alpha_2 = 0.354$.

ference between output of adjacent oscillators in the other parameters. Figure 7(a) is another type phase-inversion wave, where $\alpha_2 = \alpha_1 = \alpha_3 = 0.33$, $\lambda_1 = 0.001858$. Figure 7(b) is not a synchronization phenomenon, nor a phase-inversion wave, where $\alpha_2 = 0.3133$, $\alpha_1 = \alpha_3 = 0.33$, $\lambda_1 = 0.003962$. As shown in Fig. 7, we can understand that it is impossible to distinguish phase-inversion waves and non-phase inversion waves by the largest Lyapunov exponent. In the case where the largest Lyapunov exponent is negative, we derive the Poincare map F_p in order to con-

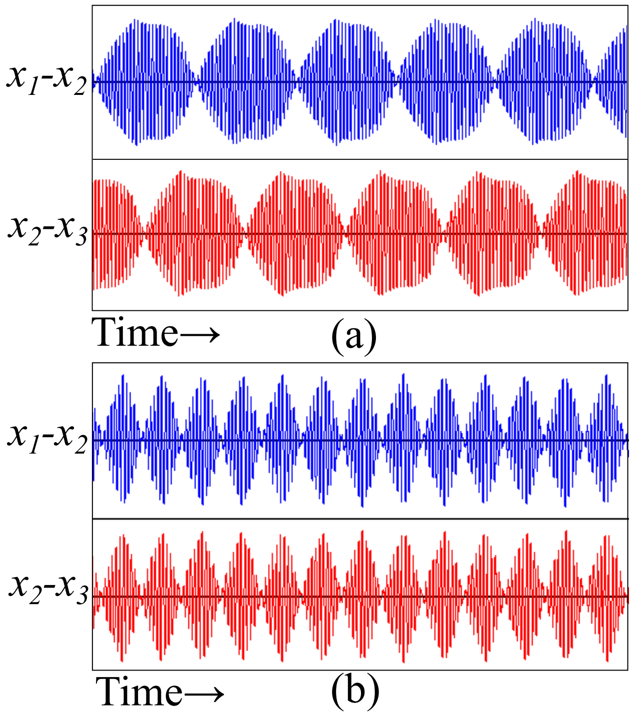


Fig. 7 The difference between output of adjacent oscillators in the other parameters.

firm the periodic motions. We define F_p as following :

$$\begin{aligned} &\mathbf{x} = {}^t(x_1, y_1, x_2, y_2, x_3, y_3), \\ &S_p = \{\mathbf{x} \mid x_2 > 0 \text{ and } y_2 = 0\}, \\ &F_p : S_p \to S_p. \end{aligned} \tag{14}$$

Figure 8 and 9 represent attractors of F_p. Parameters are same as Fig. 6. As shown in Fig. 8 and 9, we can confirm that solution is periodic in $\lambda_1 < 0$.

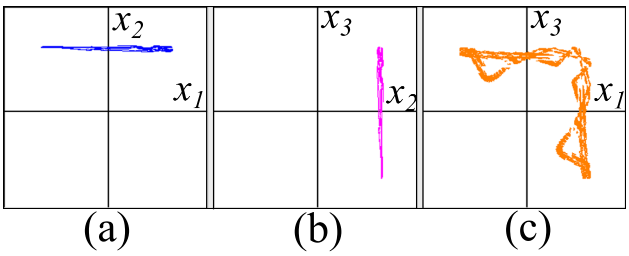


Fig. 8 Attractors of F_p in $\lambda_1 > 0$.

5. Conclusion

In this paper, we considered phase-inversion waves of three piecewise-constant oscillators coupled by hysteresis elements as a ladder. We derived the rigorous solutions in the system, and confirmed generation of phase-inversion waves. We analyzed the stability of the system by the largest Lyapunov exponents. Further, we derived Poincare maps. In the case where the largest Lyapunov exponent was negative, phase-inversion waves were not generated, and we were able to confirm the periodic motion. Our future tasks are calculating the non-largest Lyapunov exponents and analyzing in more detail by using these numbers.

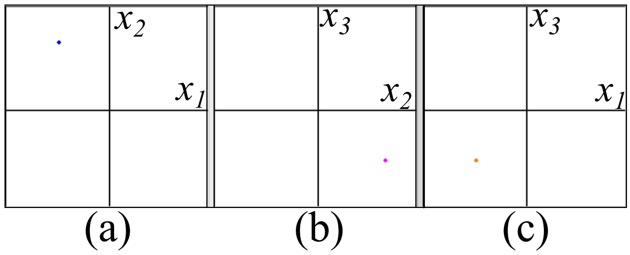


Fig. 9 Attractors of F_p in $\lambda_1 < 0$.

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