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On the study of static and Hopf bifurcation on a ring of neurons with distributed delay

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Abstract—In this paper, we study the dynamical behavior of an unidirectional ring of neurons with distributed delays affecting the communication paths. In order to detect and analyze static and Hopf bifurcations in this system, we use a version of a frequency-domain approach developed for distributed delay equations. Our results suggest that for a large enough number of neurons in the ring, a double Hopf bifurcation may also occur.

1. Introduction

Very often, specially in biology, it is convenient to use models with distributed time delays instead of concentrated ones. So, it is appropriate to consider that the evolution of the system depends on a continuum of previous values of the state variables. For example, distributed delays (DD) in biology have been used in models of cellular transmission of viruses [1], prey-predator competence in a medium [2], spread of epidemic diseases [3], etc.

In the context of neural networks, models based on DD allow to consider the multiple channels between neurons, each with a (possibly) different connection speed. Unfortunately, to deal with a model with DD is not simple, and even simple systems with just two neurons exhibit complicated dynamics [4, 5, 6]. In a previous work [7], we proposed a modified version of the method based on the Graphical Hopf Bifurcation Theorem (GHBT) [8, 9] for analyzing bifurcations in delay differential equations with DD. We showed that the delay effect can be described simply by using properties of the Laplace transform.

Several authors have studied the problem of neural networks with DD in the connection paths, and also using the GHBT. It is worth mentioning that the most common distribution considered in biological applications is the so called gamma kernel. For example, in [10] the authors analyzed a two-neuron system with a weak gamma kernel, which means an exponential decay on the weight of the past history. Later, in [5] they analyzed a similar network but considering the so called strong gamma kernel. A three-neuron network, also with a strong gamma kernel has been considered in [11]. Finally, a two-neuron system with a weak kernel, with not only delayed connections between the different neurons but also self connections in each neuron has been studied in [6]. In these articles, the authors used the chain trick to derive equivalent models expressed as ordinary differential equations. The approach that we use here is distinguished from the previous works by the fact that we do not transform the original system into an equivalent one. On the contrary, we represent the DD in the complex variable of the Laplace transform, providing a simple way to deal with the model in the frequency domain.

2. Problem formulation

In a previous article [7], we have studied a two-neuron system originally proposed in [4]. In this work, we will consider an analogous system but consisting of n neurons connected in an unidirectional ring configuration

$$\left\{\begin{array}{rcl} \dot{x}_1(t) & = & -x_1(t) + a_1 f\left[x_n(t) + b_n x_{nk}(t)\right], \\ \dot{x}_2(t) & = & -x_2(t) + a_2 f\left[x_1(t) + b_1 x_{1k}(t)\right], \\ & \vdots & \\ \dot{x}_n(t) & = & -x_n(t) + a_n f\left[x_{(n-1)}(t) + b_{n-1} x_{(n-1)k}(t)\right], \end{array}\right.$$

$$x_{ik}(t) \triangleq \int_{-\infty}^{t} k(t-u) x_i(u) du, \ i = 1, 2, ..., n. \tag{1}$$

Neuron 1 is affected by the neuron n, the neuron 2 is affected by the neuron 1 and so on, until the neuron n is affected by the neuron $n-1$. The state variables $x_i(t)$ represent the potential of each neuron, a_i determines the range of values of the variable x_i and b_i is the weight of the influence of the previous history of one neuron over the other. We will suppose that the nonlinear activation function $f(\cdot)$ is smooth and verifies the conditions $f(0) = 0$, $f' \triangleq f'(0) > 0$. The function $k(u)$ is called *kernel* and weighs the previous values of the variable $x_i(t)$. In addition, it is assumed that $k(u)$ satisfies $k(u) \geq 0, \ \forall u \geq 0$ and $\int_0^\infty k(u) du = 1$. In the following, we shall analyze the dynamics of system (1) by using the GHBT.

3. Analysis using the GHBT

System (1) can be studied through the frequency-domain approach (see [7, 9]) by considering the feedback system representation

$$\begin{cases} \dot{\mathbf{x}}(t) &= A\mathbf{x}(t) + B\mathbf{g}(\mathbf{y}(t), \mathbf{y}_k(t); \mu) \\ \mathbf{y}(t) &= -C\mathbf{x}(t), \quad \mathbf{y}_k(t) = -C\mathbf{x}_k(t) \end{cases} \quad (2)$$

where $\mathbf{x} = (x_1, ..., x_n)^T$, $A = -I_n$, $B = C = I_n$ and

$$\mathbf{g}(\mathbf{y}(t), \mathbf{y}_k(t); \mu) = \begin{pmatrix} a_1 f(-b_n y_{nk} - y_n) \\ a_2 f(-b_1 y_{1k} - y_1) \\ \vdots \\ a_n f(-b_{(n-1)} y_{(n-1)k} - y_{(n-1)}) \end{pmatrix}, \quad (3)$$

where I_n is the identity matrix of order n and μ represents the vector of parameters. Although it is obvious that the equilibrium equations

$$\widehat{x}_{(k+1)mod\, n} = a_{(k+1)mod\, n} f\left[\widehat{x}_k(1 + b_k)\right], \quad 1 \le k \le n, \quad (4)$$

may have more than one solution, we will focus on the dynamics around the trivial equilibrium $\widehat{\mathbf{y}} = 0$. System (2) is equivalent to a linear subsystem with a nonlinear feedback $\mathbf{g}(\cdot)$. If we consider $K(s)$, the Laplace transform[1] of the kernel function $k(t)$, the linear subsystem is represented by the transfer function

$$G(s) = C(sI_n - A)^{-1} B \begin{pmatrix} I_n \\ I_n\, K(s) \end{pmatrix} = \frac{1}{s+1} \begin{pmatrix} I_n \\ I_n K(s) \end{pmatrix}. \quad (5)$$

In order to study the stability properties of the equilibrium $\widehat{\mathbf{y}} = 0$, we compute the $n \times 2n$ Jacobian matrix

$$J(\mu) = \left(\frac{\partial \mathbf{g}}{\partial \mathbf{y}} \;\middle|\; \frac{\partial \mathbf{g}}{\partial \mathbf{y}_k} \right)\Bigg|_{\widehat{\mathbf{y}}^* = \mathbf{0}} \quad (6)$$

and the characteristic equation in the frequency domain $|\lambda I_{2n} - G^*(s) J(\mu)| = 0$ becomes

$$h(\lambda, s; \mu) = \frac{\lambda^n}{(s+1)^n} \{\lambda^n (s+1)^n - (-1)^n \delta^n \psi(s)\} = 0, \quad (7)$$

where $\delta^n \triangleq (f')^n \prod_{i=1}^{n} a_i$, $\psi(s) \triangleq \prod_{i=1}^{n} \eta_i(s)$ and $\eta_i(s) \triangleq [b_i K(s) + 1]$.

3.1. Static bifurcations

As can be seen in [9], the bifurcations of equilibria can be detected in the parameter space by looking for solutions of $h(-1, 0, \mu) = 0$.

3.1.1. Case 1: Different weights for every neuron

Let us first suppose that the weights of the previous history can be different for every neuron. Then we have

[1] Notice that under the conditions that we have assumed for the kernel function, $K(s)$ exists at least for $\mathrm{Re}(s) > 0$.

Proposition 3.1.1 *System (1) exhibits a static bifurcation (ST) or bifurcation of equilibria only if the combination of parameters satisfy*

$$1 = \delta^n \psi(0). \quad (8)$$

Proof: It is enough to consider $h(-1, 0, \mu) = 0$, which is equal to $(-1)^n - (-1)^n \delta^n \psi(0) = 0$. □

Remark 3.1.2 *The system:*

- *Verifies a necessary condition for a ST if the previous history has an excitatory effect on every neuron ($b_i > 0, \forall i$).*
- *Will not exhibit a ST if $b_i = -1$, for some i.*
- *May exhibit a ST only if $\psi(0) > 0$ and $b_i \neq -1, \forall i$, when the previous history has an excitatory effect on some neurons.*

Proposition 3.1.3 *System (1) exhibits a double zero (DZ) bifurcation only if the mean delay $\tau_m \triangleq \int_0^\infty uk(u)du$ satisfies*

$$\tau_m = -\frac{n}{\Delta_1}, \quad \Delta_1 \triangleq \sum_{i=1}^{n} \frac{b_i}{(b_i + 1)}. \quad (9)$$

Proof: Notice that $h(\lambda, s; \mu) = 0 \Leftrightarrow \widetilde{h}(\lambda, s; \mu) = 0$ where $\widetilde{h}(\lambda, s; \mu) \triangleq \lambda^n (s+1)^n - (-1)^n \delta^n \psi(s)$. Then we can find

$$\left.\frac{\partial \widetilde{h}}{\partial s}\right|_{(-1,0;\mu)} = (-1)^n \left[n - \delta^n K'(0) \psi(0) \sum_{i=1}^{n} \frac{b_i}{b_i + 1} \right]. \quad (10)$$

The condition for the existence of a DZ bifurcation is obtained when the right-hand side of Eq. (10) is equal to zero, and simultaneously, the ST condition $1 = \delta^n \psi(0)$ given by (8) holds. Taking into account that $K'(0) = -\int_0^\infty uk(u)du = -\tau_m$, we arrive to (9). □

Remark 3.1.4 *The system is likely to exhibit a DZ bifurcation only if the quantity Δ_1 is negative.*

Remark 3.1.5 *If $k(t) = \delta(t - \tau)$, where $\delta(\cdot)$ is the Dirac impulse, then the delay is concentrated (point delay) and in this case τ should satisfy (9) in order to produce a DZ bifurcation in system (1).*

Proposition 3.1.6 *System (1) will exhibit a triple zero (TZ) bifurcation only if the following conditions hold*

$$\begin{array}{lll} \text{(ST)} & -1 + \delta^n \psi(0) = 0 & \psi(0) > 0 \\ \text{(DZ)} & n + \tau_m \Delta_1 = 0 & \Delta_1 < 0 \\ \text{(TZ)} & -\tau_m f(\tau_m) + \sigma^2 = 0 & \Delta_2 > 0, \; \tau_m > \tau_m^{(1)}, \end{array} \quad (11)$$

where

$$\Delta_2 = \sum_{i=1}^{n-1} \frac{b_i b_{i+1}}{(b_i+1)(b_{i+1}+1)}, \quad f(\tau_m) = \frac{2\Delta_2}{n} \tau_m^2 - \tau_m - (n-1)$$

$$\sigma^2 = \int_0^\infty (t - \tau_m)^2 k(t) dt \quad \tau_m^{(1)} = \frac{n}{4\Delta_2} \left\{ 1 + \sqrt{1 + \frac{8(n-1)\Delta_2}{n}} \right\}.$$

Proof: After some calculations, we find

$$\frac{\partial^2 \widetilde{h}}{\partial s^2} = n(n-1)\lambda^n (s+1)^{n-2} - (-1)^n \psi \delta^n \left\{ 2\sum_{i=1}^{n-1} \frac{\eta_i' \eta_{i+1}'}{\eta_i \eta_{i+1}} + \sum_{i=1}^{n} \frac{\eta_i''}{\eta_i} \right\}, \tag{12}$$

and the condition $\partial^2 h/\partial s^2|_{(\lambda,s;\,\mu)=(-1,0;\mu)} = 0$ becomes

$$n(n-1) - \delta^n \psi(0) \left\{ 2\tau_m^2 \Delta_2 + K'' \Delta_1 \right\} = 0, \tag{13}$$

where $K'' \triangleq K''(0) = \int_0^\infty t^2 k(t) dt$. Then we have

$$n(n-1) - \delta^n \psi(0) \left\{ 2\tau_m^2 \Delta_2 + (\sigma^2 + \tau_m^2) \Delta_1 \right\} = 0. \tag{14}$$

Replacing the conditions for a ST (8) and DZ (9) into (14), we arrive to

$$n(n-1) - \left\{ 2\tau_m^2 \Delta_2 + (\sigma^2 + \tau_m^2)(-n/\tau_m) \right\} = 0, \tag{15}$$

which leads to $\sigma^2 = \tau_m f(\tau_m)$. In order to have a solution, Δ_2 should be positive and obviously the mean delay must satisfy $f(\tau_m) > 0$. The leftmost root of $f(\tau_m)$ is negative and the rightmost root of $f(\tau_m)$ is positive. Then, in order to have a TZ, the mean delay must satisfy $\tau_m > \tau_m^{(1)}$. □

3.1.2. *Case 2: Same weight for every neuron*

Now let us consider that the weights of the previous history is the same for every neuron, *i.e.*, $b_1 = b_2 = ... = b_n = b$. Let us call the neural network (1) *symmetrical* in this case (notice that the neurons are not necessarily identical). Then we have

$$\widetilde{h}(\lambda, s; \mu) \triangleq \lambda^n (s+1)^n - (-1)^n \delta^n \left[bK(s) + 1 \right]^n, \tag{16}$$

and the condition for a ST bifurcation reduces to $1 = \delta^n (b + 1)^n$.

Remark 3.1.7 *The symmetrical system will be likely to exhibit a ST if*

- *n is even.*
- *n is odd and δ and $b + 1$ have the same sign.*

Analogously, after some calculations we can state the condition for a DZ bifurcation as

$$1 + \gamma \tau_m = 0, \qquad \gamma \triangleq \frac{b}{b+1}. \tag{17}$$

Remark 3.1.8 *In the symmetrical case, the system can exhibit a DZ bifurcation only if $\gamma < 0$, i.e., $0 < b < 1$.*

By computing the second derivative $\partial^2 \widetilde{h}/\partial s^2$ evaluted at $(\lambda, s; \mu) = (-1, 0; \mu)$ it is easy to show that the TZ bifurcation can not occur in the symmetrical case.

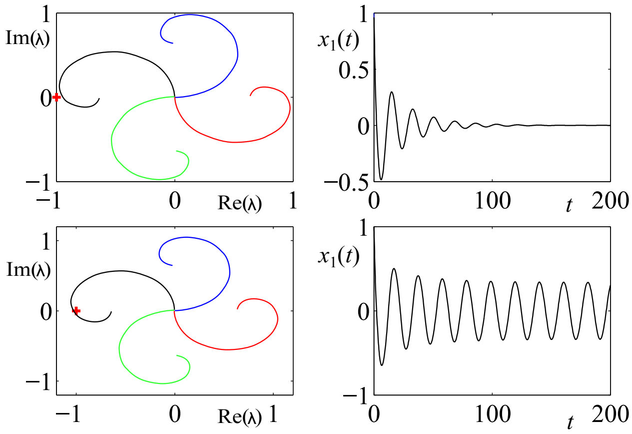

Figure 1: Branches of the Nyquist diagram (λ_0 in red, λ_1 in blue, λ_2 in black and λ_3 in green) and the corresponding numerical simulations for $n = 4$, with $a_i = 1.275$, $b_i = 0.5$, $\forall i$, $p = 1$. Up: $\alpha = 0.35$, and the equilibrium point at the origin is stable. Down: $\alpha = 0.25$, the equilibrium point is unstable and a stable periodic solution exists.

3.2. Hopf Bifurcations

The characteristic equation (7) can be solved for the variable λ as

$$\widehat{\lambda}(s; \mu) = \sqrt[n]{\frac{(-1)^n \delta^n \psi(s)}{(s+1)^n}}, \tag{18}$$

then we have n solutions (branches), and the corresponding frequency plots can be obtained from

$$\widehat{\lambda}_k(i\omega; \mu) = \left| \frac{|\delta| \, |\psi(i\omega)|}{|1 + i\omega|} \right|^{1/n} e^{i(\phi + 2\pi k)/n}, \quad k = 0, 1, ..., n-1, \tag{19}$$

where

$$\phi = \begin{cases} Arg\{\psi(i\omega)\} - n \arctan(\omega), & n \text{ even, or } n \text{ odd and } \delta < 0, \\ Arg\{\psi(i\omega)\} - n \arctan(\omega) + \pi, & n \text{ odd and } \delta > 0. \end{cases} \tag{20}$$

In a similar way, for the symmetrical case, the characteristic solutions are given by

$$\widehat{\lambda}_k(i\omega; \mu) = -\delta \frac{[bK(i\omega) + 1]}{1 + i\omega} e^{i2\pi k/n}, \quad k = 1, ..., n-1. \tag{21}$$

Figure 1 shows the Hopf bifurcation appearing for the case of 4 neurons, where the kernel function is a weak gamma kernel $k(t) = \alpha e^{-\alpha t}$, $t \geq 0$, then $K(s) = \alpha/(s + \alpha)$. For $\alpha = 0.35$, none of the branches of the Nyquist plot encloses the critical point -1, and the equilibrium point is stable. For $\alpha = 0.25$, one of the branches of the Nyquist curve encircles the critical point and so the equilibrium is unstable and a stable periodic solution exists. As (19) has n solutions, it is feasible that for a large enough number of neurons, more than one branch of the Nyquist plot may may pass through the point -1. For example, Fig. 2 shows the corresponding Nyquist diagrams for $n = 15$. Notice that there are

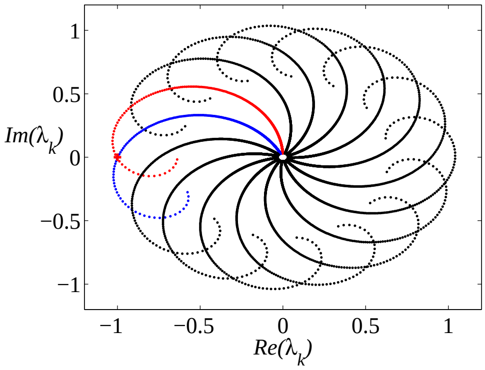

Figure 2: Nyquist diagrams for $n = 15$, with $a_i = 1.275$, $b_i = 0.5$, $\forall i$, $p = 1$ and $\alpha = 0.27$. Notice that there are two branches passing through the critical point $-1 + 0i$, $\lambda_0(\mathrm{i}\omega)$ (red) and $\lambda_1(\mathrm{i}\omega)$ (blue).

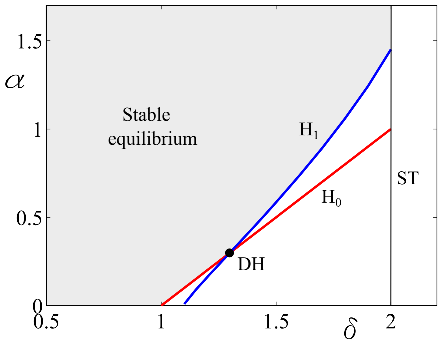

Figure 3: Hopf bifurcation curves in the (δ, α) parameter space. The red curve H_0 is generated by the eigenvalue $\widehat{\lambda}_0(\mathrm{i}\omega)$ and the blue curve H_1 is generated by $\widehat{\lambda}_1(\mathrm{i}\omega)$.

two characteristic plots passing through the critical point for the parameter values indicated in the figure. Moreover, varying the parameters α and δ we can continue the Hopf curves in the parameter space, as shown in Fig. 3. In this diagram, H_0 and H_1 indicate the Hopf curves provoked by the characteristic solutions $\widehat{\lambda}_0(i\omega)$ and $\widehat{\lambda}_1(i\omega)$, respectively. The Hopf curves intersect each other on a double Hopf bifurcation point (noted as DH in Fig. 3). Both Hopf curves end at the vertical line corresponding to ST bifurcation.

4. Conclusions

We analyzed a ring of neurons affected by DD in the connection paths. Provided by the effect of the DD can be represented simply using properties of the Laplace transform, it is straightforward to apply the GHBT for such a kind of models. We have shown how to detect static and dynamic bifurcations, whose occurrence is related with the properties of the delay distribution, as the mean and standard deviation.

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