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Chaotic attractors in a novel fractional-order memristor-based system

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Abstract– In this paper a novel fractional system including a memristor is introduced. Specifically, chaotic behaviors in the simplest fractional memristor-based system are shown. Stability analysis of the system equilibria and numerical integrations are carried out, with the aim to show that chaos can be found when the fractional-order of the proposed system is as low as 2.895.

1. Introduction

Fractional calculus represents a topic of great interest in the field of nonlinear theory and its applications [1]. This happens because several physical phenomena can be described more accurately by fractional differential equations rather than integer models [1]. Nowadays a number of techniques are available for approximating fractional derivatives and integrals [2]. Therefore, fractional calculus plays an important role in physics, electrical circuit theory, control systems and chemical mixing [3]. In particular, a significant role is played in chaos theory, where it has been shown that chaotic phenomena can be obtained in nonlinear systems with fractional-order [4]. Referring to chaotic dynamics, several fractional systems have been proposed starting from the chaotic integer counterparts. For example, by considering the Chua's circuit, some fractional counterparts (analyzed using different numerical methods) have been proposed [5]. Similarly, the fractional Rössler system, the fractional Lorenz-based multi-wing system and the fractional Chen system have been proposed starting from the corresponding integer systems [6]-[7]. Recently, an interesting example of fractional memristor-based chaotic system has been introduced, starting from the integer counterpart [8]. The *memristor* is the missing circuit element studied by Chua in 1971 and realized by Williams's group of HP Labs in 2008 [9].

Based on previous considerations, the idea of developing a new fractional system arose. Specifically, in this paper the simplest fractional memristor-based chaotic system is proposed starting from the integer counterpart [10]. A theoretical analysis of its dynamics is illustrated, along with numerical simulations, showing chaos.

The paper is organized as follows. In Section 2 the fundamentals of fractional derivative and memristor are illustrated. In Section 3 a stability analysis for the equilibrium point of the proposed fractional system is carried out. Then, a condition is considered, according to which the fractional system remains chaotic starting from the chaotic integer counterpart. In Section 4 a predictor-corrector algorithm is applied to solve the considered system. In particular, a chaotic attractor is found when the order of the derivative is $q = 0.965$.

2. Basic notions on fractional calculus and memristor

2.1. Caputo's fractional derivative

Referring to fractional calculus, among the different definitions of derivative proposed in the literature, in this work the fractional differential operator ${}^{*}D_{t_0}^{q}$ of order $q \in \Re^{+}$ proposed by Caputo is utilized [11]:

$$
{}^{*}D_{t_0}^{q}x(t) = J_{t_0}^{m-q}D_{t_0}^{m}x(t) = \begin{cases} \dfrac{1}{\Gamma(m-q)}\displaystyle\int_{t_0}^{t}\dfrac{x^{(m)}(\tau)}{(t-\tau)^{q+1-m}}\,d\tau, \\ \qquad \text{for} \quad m-1 < q < m \\ \dfrac{d^m}{dt^m}x(t), \qquad \text{for} \quad q = m \end{cases} \tag{1}
$$

where $\Gamma(q)$ is the gamma function whereas $m \in N$. Two properties of the operator ${}^{*}D_{t_0}^{q}$ are ${}^{*}D_{t_0}^{q}J_{t_0}^{q}x(t) = x(t)$ and

$$
J_{t_0}^{q}({}^{*}D_{t_0}^{q})x(t) = J_{t_0}^{q}J_{t_0}^{m-q}D_{t_0}^{m}x(t) = x(t) - \sum_{k=0}^{m-1}x^{(k)}(t_0^{+})\frac{(t-t_0)^k}{k!}.
$$

Note that the Caputo derivative (1) is a better choice for fractional-order circuits, since the initial values required by the Caputo definition coincide with identifiable physical states in the considered system [2].

2.2. Memristor model

The memristor realized by a team from HP Company is a electronic device that has generated great interest by virtue of the large number of its potential applications (i.e., applications to next generation computers and brain-like "neural" computers) [9]. Given the charge Q and the magnetic flux Φ, the memristor is a passive two-terminal device generally described by the nonlinear *constitutive relation* $v_M = M(Q)i_M$ between the terminal voltage v_M and current i_M [12]. The nonlinear *memristance* $M(Q)$ is defined by $M(Q) = d\Phi(Q)/dQ$, representing the slope of the function $\Phi = \Phi\ (Q)$. Note that a memristor with a differentiable $Q-\Phi$ characteristic curve is passive if, and only if, its memristance $M(Q)$ is non-negative [12], i.e. $M(Q) \triangleq \left(d\Phi(Q)/dQ\right) \geq 0$. Among the different models of memristors proposed in the literature, this paper focuses on the *current-controlled memristive system* described by the circuit equations [12]:

$$
\begin{cases} v_M = R(u)\ i_M \\ \dot{u} = f(u, i_M) \end{cases} \tag{2}
$$

where u is the internal state, $f(u, i_M)$ is the internal state function whereas $R(u)$ is the memristance.

3. The fractional-order memristor-based system

Among the various integer-order memristor systems in literature [13]-[15], the one proposed in [10] is the simplest autonomous integer chaotic circuit that includes a memristor. Namely, the circuit in [10] contains only three elements connected in series, i.e. an inductor with current i_L, a capacitor with voltage v_C and a memristor with memristance $R(u) = 1.5(u^2 - 1)$. According to [10], where the internal state is $f(u, i_M) = i_M - 0.6u - i_M u$, the *integer-order memristor* equations (2) can be rewritten as:

$$\begin{cases} \dot{v}_C = i_L \\ \dot{i}_L = -\frac{1}{3} v_C - \frac{1}{2}(u^2 - 1)\, i_L \\ \dot{u} = -i_L - \frac{3}{5} u + i_L u \end{cases} \tag{3}$$

By considering the equations (3) and by defining $x = v_C$, $y = i_L$ and $z = u$, the equations of the proposed *fractional memristor-based system* are derived:

$$\begin{cases} {}^{*}D^q x = y \\ {}^{*}D^q y = -\frac{1}{3}\left(x + \frac{3}{2}(z^2 - 1) y \right) \\ {}^{*}D^q z = -y - \frac{3}{5} z + yz \end{cases} \tag{4}$$

where ${}^{*}D^q$ denotes the Caputo fractional operator defined in (1) with initial time $t_0 = 0$ and order $q \in (0, 1)$ [11].

4. Dynamics of the simplest fractional memristor-based system

In order to study system (4), at first a stability analysis of its equilibrium points is developed. Then, a condition for system (4) to remain chaotic is considered.

4.1. Stability results

By defining the vector $\boldsymbol{x} = [x \quad y \quad z]^T$, system (4) can be rewritten in the following general nonlinear form:

$${}^{*}D^q \boldsymbol{x} = \boldsymbol{f}(\boldsymbol{x}). \tag{5}$$

Note that system (4)-(5) has *only one equilibrium point* at the origin $\boldsymbol{x}_{eq} = [0 \quad 0 \quad 0]^T$. According to Lyapunov's indirect method, the stability property of $\boldsymbol{x}_{eq} = \boldsymbol{0}$ for system (5) can be investigated by considering its stability for the corresponding linear system. To this purpose, the nonlinear fractional system (5) is linearized around the equilibrium point $\boldsymbol{x}_{eq} = \boldsymbol{0}$ as:

$${}^{*}D^q \boldsymbol{x} = \boldsymbol{J}\, \boldsymbol{x}\ , \quad (0 < q < 1) \tag{6}$$

where $\boldsymbol{J} = \left(\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}} \right)_{\boldsymbol{x}_{eq} = \boldsymbol{0}}$ is the Jacobian matrix of $\boldsymbol{f}$ with respect to $\boldsymbol{x}$ at $\boldsymbol{x}_{eq} = \boldsymbol{0}$. Now, for the linear fractional system (6) the following stability theorem is given [16].

Theorem 1. The linear fractional system (6) is asymptotically stable if all the eigenvalues λ of the Jacobian matrix $\boldsymbol{J}$ satisfy the condition:

$$|\arg(\lambda)| > q\frac{\pi}{2}.\square \tag{7}$$

Now, the following result, based on Lyapunov's direct method, states the stability relationship between the fractional linear system (6) and the fractional nonlinear system (5).

Theorem 2. If the fractional linearized system (6) is asymptotically stable (i.e., if all eigenvalues λ of the Jacobian matrix $\boldsymbol{J}$ satisfy $|\arg(\lambda)| > q\frac{\pi}{2}$), then the equilibrium point $\boldsymbol{x}_{eq} = \boldsymbol{0}$ of the fractional nonlinear system (5) is asymptotically stable. □

The proof of Theorem 2 can be found in [17]. Based on previous theorems, the following Corollary on the unstable equilibria is given.

Corollary. If at least one eigenvalue λ of the Jacobian matrix $\boldsymbol{J}$ satisfy $|\arg(\lambda)| < q\frac{\pi}{2}$, then the equilibrium $\boldsymbol{x}_{eq} = \boldsymbol{0}$ of the fractional nonlinear system (5) is unstable. □

Previous theoretical results make clear that the stability condition for fractional systems differs from the condition given for integer systems. In particular, the right half plane (unstable region) for integer systems maps into a 'wedge' in the case of fractional systems, indicating that the unstable region becomes smaller and smaller when the value of order q is decreased (see Fig.1).

4.2. A condition to remain chaotic

Based on previous results, now the aim is to apply to system (4) the necessary condition given in [18], according to which a fractional system, derived from the *chaotic* integer counterpart, remains *chaotic* when q is larger than a proper value. By considering both systems (3) and (4), it can be readily shown that the Jacobian matrix $\boldsymbol{J}$ associated to $\boldsymbol{x}_{eq} = \boldsymbol{0}$ has one negative real eigenvalue and two complex conjugate eigenvalues with positive real parts:

$$\begin{cases} \lambda_1 = -0.6000 \\ \lambda_2 = \alpha + j\beta = 0.2500 + j0.5204 \\ \lambda_3 = \alpha - j\beta = 0.2500 - j0.5204 \end{cases}. \tag{8}$$

Recall that an equilibrium point is said a saddle point of index 2 if it has one stable eigenvalue and two complex conjugate unstable eigenvalues [18]. According to the previous considerations, it is worth noting that for the integer system (3) the equilibrium $\boldsymbol{x}_{eq} = \boldsymbol{0}$ is undeniably a saddle point of index 2, whereas for the fractional system (4) the equilibrium $\boldsymbol{x}_{eq} = \boldsymbol{0}$ is a saddle point of index 2 only if q is selected so that λ_2 and λ_3 lie in the unstable region (see Fig.1). Now assume that the integer system (3) displays a chaotic attractor around the saddle point $\boldsymbol{x}_{eq} = \boldsymbol{0}$ of index 2. According to [18], a necessary condition for the fractional system (4) to remain chaotic is keeping the eigenvalues λ_2 and λ_3 in the unstable region.

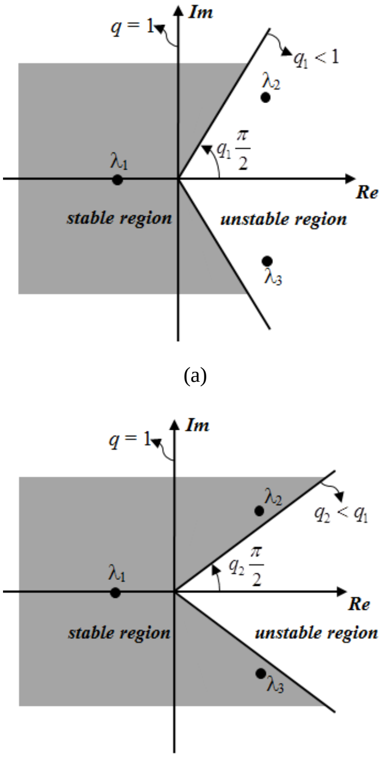

Fig.1. Examples of stable and unstable regions in fractional systems for two choices of q. (a): eigenvalues λ_2 and λ_3 belong to the *unstable region* for a selected $q_1 < 1$; (b): the same eigenvalues λ_2 and λ_3 now belong to the *stable region* for a selected $q_2 < q_1 < 1$.

This means that the order q must satisfy the condition [18]:

$$q > \frac{2}{\pi}\tan^{-1}\left(\frac{|\beta|}{\alpha}\right). \tag{9}$$

The inequality (9) can be derived from the previous Corollary. Namely, according to the Corollary, the eigenvalues λ_2 and λ_3 are kept in the unstable region by taking $|\arg(\lambda_{2/3})| < q\frac{\pi}{2}$, which is equivalent to $\left|\tan^{-1}(\beta/\alpha)\right| < q(\pi/2)$. Since $\alpha > 0$, it follows that $\tan^{-1}(|\beta|/\alpha) < q(\pi/2)$, from which the condition (9) is readily derived. In conclusion, given the eigenvalues (8) where $\alpha = 0.2500$ and $\beta = 0.5204$, the condition (9) for the fractional system (4) is satisfied when $q > 0.715$.

5. Chaotic attractor

The predictor-corrector algorithm, which belongs to the Adams–Bashforth–Moulton schemes, is adopted to solve the fractional system (4) [19]. By applying the predictor-corrector algorithm, the solution of the fractional system (4) is:

$$\begin{cases} x_h(t_{n+1}) = x_0 + \frac{h^q}{\Gamma(q+2)}\left(y^{(p)}(t_{n+1})\right) + \\ \quad + \frac{h^q}{\Gamma(q+2)}\sum_{j=0}^{n}\alpha_{j,n+1}\left(y(t_j)\right) \\ y_h(t_{n+1}) = y_0 + \frac{h^q}{\Gamma(q+2)}\left(-\frac{1}{3}x^{(p)}(t_{n+1}) + \frac{1}{2}y^{(p)}(t_{n+1}) - \frac{1}{2}y^{(p)}(t_{n+1})z^{2(p)}(t_{n+1})\right) + \\ \quad + \frac{h^q}{\Gamma(q+2)}\sum_{j=0}^{n}\alpha_{j,n+1}\left(-\frac{1}{3}x(t_j) + \frac{1}{2}y(t_j) - \frac{1}{2}y(t_j)z^2(t_j)\right) \\ z_h(t_{n+1}) = z_0 + \frac{h^q}{\Gamma(q+2)}\left(-y^{(p)}(t_{n+1}) - \frac{6}{10}z^{(p)}(t_{n+1}) + y^{(p)}(t_{n+1})z^{(p)}(t_{n+1})\right) + \\ \quad + \frac{h^q}{\Gamma(q+2)}\sum_{j=0}^{n}\alpha_{j,n+1}\left(-y(t_j) - \frac{6}{10}z(t_j) + y(t_j)z(t_j)\right) \end{cases} \tag{10}$$

in which the predicted variables are:

$$\begin{cases} x^{(p)}(t_{n+1}) = x_0 + \frac{1}{\Gamma(q)}\sum_{j=0}^{n}\beta_{j,n+1}\left(y(t_j)\right) \\ y^{(p)}(t_{n+1}) = y_0 + \frac{1}{\Gamma(q)}\sum_{j=0}^{n}\beta_{j,n+1}\left(-\frac{1}{3}x(t_j) + \frac{1}{2}y(t_j) - \frac{1}{2}y(t_j)z^2(t_j)\right) \\ z^{(p)}(t_{n+1}) = z_0 + \frac{1}{\Gamma(q)}\sum_{j=0}^{n}\beta_{j,n+1}\left(-y(t_j) - \frac{6}{10}z(t_j) + y(t_j)z(t_j)\right) \end{cases} \tag{11}$$

where

$$\alpha_{j,n+1} = \begin{cases} n^{q+1} - (n-q)(n+1)^q, & j = 0 \\ (n-j+2)^{q+1} + (n-j)^{q+1} - 2(n-j+1)^{q+1}, & 1 \le j \le n \\ 1 & j = n+1 \end{cases} \tag{12}$$

and

$$\beta_{j,n+1} = \frac{h^q}{q}\left((n+1-j)^q - (n-j)^q\right), \quad 0 \le j \le n. \tag{13}$$

By taking into account the results illustrated in the previous Section, at first the integer system (3) is considered. In [10] it has been shown that the integer-order system (3) displays a chaotic attractor around the saddle point $\boldsymbol{x}_{eq}$ = **0** of index 2 (see eq. (8)), which is reported in Fig.2.

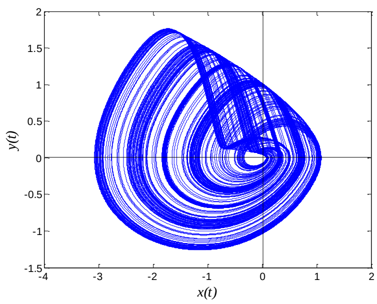

Fig.2. Chaotic attractor of the integer system (3) on the (x,y)-plane.

Now, according to the inequality (9), the fractional system (4) satisfies the *necessary* condition for remaining chaotic when $q > 0.715$. Figure 3 shows the plot of a chaotic attractor found for $q = 0.965$. Note that the chaotic nature of the attractor is confirmed by the computation of the maximum Lyapunov exponent [20]. Since a positive value is found ($\lambda_{\max}$ = 0.969), this confirms the chaotic dynamics of system (4).

Now, according to Theorem 2, for q < 0.715 the origin $\boldsymbol{x}_{eq} = \boldsymbol{0}$ of system (4) is an asymptotically stable equilibrium point (i.e., all its eigenvalues lie in the stable/grey region of Fig.1(b)). In this regard, Figure 4 depicts the system behavior for q = 0.710, indicating that the stable dynamics start from the initial condition (0.1, -0.1, 0.2) and reach the origin.

Finally, it worth noting that the condition (9) is not *sufficient* for assuring that system (4) remains chaotic. To this purpose, for q = 0.720 (i.e. q > 0.715) fractional system (4) possesses a non-chaotic behaviour (see Fig.5).

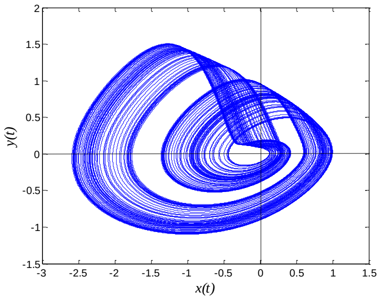

Fig.3. Chaotic attractor of the fractional system (4) when $q = 0.965$.

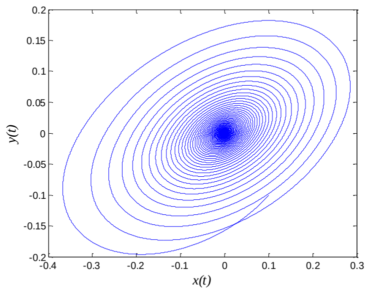

Fig.4. Stable dynamics for the fractional system (4) when $q = 0.710$.

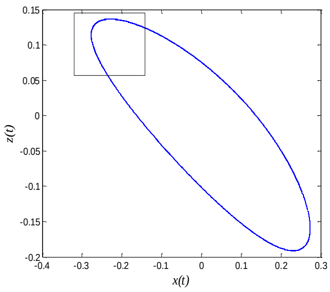

Fig.5. Non-chaotic behaviour of the fractional system (4) for $q = 0.720$.

6. Conclusions

In this paper, the simplest fractional-order memristor-based system characterized by chaotic behaviors has been presented. A theoretical analysis of the system dynamics has been illustrated in detail. In particular, a stability analysis for the equilibrium point of the proposed fractional system has been carried out. Finally, accurate numerical simulations via the predictor-corrector algorithm have shown the presence of a chaotic attractor obtained when q = 0.965.

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