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First example of hyperchaos in fractional-order systems without equilibrium points

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Abstract– The presence of chaos in *fractional-order systems without equilibrium points* represents a recent challenging topic in nonlinear dynamics. On the other hand, to the best of authors' knowledge, the presence of *hyperchaos* has not been found in these systems. This paper bridges the gap by introducing a new example of fractional hyperchaotic system without equilibrium points. The conducted analysis has shown that hyperchaos exists in the proposed system when its order is as low as 3.84. An interesting application of hyperchaotic synchronization to the considered fractional system is provided.

1. Introduction

During the last decades, researchers have found out that problems encountered in viscoelasticity, electromagnetic waves, quantitative finance, electrical circuit theory and control systems can be more accurately described using fractional calculus [1-3]. More recently, great attention has been focused on *chaotic* (only one positive Lyapunov exponent) and *hyperchaotic* (two or more positive Lyapunov exponent) behaviors of nonlinear fractional-order systems [4-11]. Some examples include the fractional chaotic Chua's circuit [4-5], the fractional chaotic Lorenz system [6], the fractional chaotic Chen system [7-8], the fractional hyperchaotic Rössler system [9] and the fractional systems generating multi-scroll and multi-wing attractors [10-11]. Note that all these fractional systems are characterized by *one or more equilibrium points*. However, a very challenging topic is the study of fractional-order systems *without equilibrium points*. Namely, the presence of chaos in nonlinear systems without equilibria is very surprising since they can have neither homoclinic nor heteroclinic orbits [12], and thus the Shilnikov theorem cannot be applied [13]. In this regard, referring to the presence of *chaos in fractional systems with no equilibria*, only very few papers have been published [14-16]. On the other hand, referring to the presence of *hyperchaos in fractional systems with no equilibria*, to the best of our knowledge, no paper has been published in the literature so far.

Based on these considerations, this paper aims to bridge the gap by introducing a new example of fractional hyperchaotic system with no equilibria. The conducted analysis has shown that the proposed system exhibits *hyperchaotic attractors when the system order is as low as 3.84*. An application of hyperchaotic synchronization to the considered fractional system is also illustrated.

The paper is organized as follows. In Section 2 the fundamentals of fractional calculus and the predictor-corrector method are reported. In Section 3 the considered fractional-order system with no equilibria is studied. An attractor is found when the order of the derivative is q = 0.96 and its hyperchaotic nature is confirmed by the application of a recent numerical method [17]. Finally, in Section 4 an example of synchronization involving the considered hyperchaotic fractional system is described.

2. Theoretical background

The *Riemann-Liouville fractional integral operator* $J_{t_0}^q$ of order $q \in \Re^+$ is defined on the Lebesque space $L_1[t_0, t_1]$ by

$$J_{t_0}^q x(t) = \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-\tau)^{q-1} x(\tau) d\tau \,, \tag{1}$$

where $\Gamma(q)$ is the Gamma function [18]. In this manuscript the *Caputo differential operator* ${}^{*}D_{t_0}^q$ is utilized:

$${}^{*}D_{t_0}^q x(t) = J_{t_0}^{m-q} D_{t_0}^m x(t) = \frac{1}{\Gamma(m-q)} \int_{t_0}^{t} \frac{x^{(m)}(\tau)}{(t-\tau)^{q+1-m}} d\tau. \tag{2}$$

where $m-1 < q \le m$ and $m \in N$ (i.e., $m = \text{ceil}(q)$) [19].

Based on the Caputo's definition (2), the following form of fractional-order differential equation is considered:

$${}^{*}D_{0,t}^q x(t) = f(x(t)), \quad x(0) = x_0, \quad q \in (0,1) \,. \tag{3}$$

It has been demonstrated that the initial value problem (3) is equivalent to a Volterra integral equation [20],

$$x(t) = x_0 + \frac{1}{\Gamma(q)} \int_0^t (t-\tau)^{q-1} f(\tau, x(\tau)) d\tau \,. \tag{4}$$

Now, the equation (4) is solved by applying the *predictor-corrector* iterative algorithm, which belongs to the Adams–Bashforth–Moulton (ABM) schemes. By taking $0 \le t \le T$ and by setting $h = T/N$ ($N \in \mathbb{Z}^+$), $t_n = nh$, n = 0,1,..., N, equation (4) can be discretized as [20]:

$$x(t_{n+1}) = x_0 + \frac{h^q}{\Gamma(q+2)} \sum_{j=0}^{n+1} \alpha_{j,n+1} f(t_j, x(t_j)) \,, \tag{5}$$

$$\alpha_{j,n+1} = \begin{cases} n^{q+1} - (n-q)(n+1)^q, & j = 0 \\ (n-j+2)^{q+1} + (n-j)^{q+1} - 2(n-j+1)^{q+1}, & 1 \le j \le n \\ 1 & j = n+1 \end{cases} \tag{6}$$

Equation (5) can be rewritten as [20]:

$$x(t_{n+1}) = x_0 + \frac{h^q}{\Gamma(q+2)} f(t_{n+1}, x(t_{n+1})) + + \frac{h^q}{\Gamma(q+2)} \sum_{j=0}^{n} \alpha_{j,n+1} f(t_j, x(t_j)) . \qquad (7)$$

The solution via the ABM method is carried out by first *predicting* $x(t_{n+1})$ using the *explicit* Adams-Bashforth formula (obtaining the prediction $\hat{x}(t_{n+1})$) and then *correcting* (obtaining the final value $x(t_{n+1})$) [20].

3. A new hyperchaotic fractional system with no equilibria

Very recently, in [21] the first example of a 4-D *integer-order* hyperchaotic system *with no equilibria* was given. The system proposed in [21] possesses no characteristics such as pitchfork bifurcation, Hopf bifurcation, and so on. The presence of hyperchaos in such system is very surprising since it can have neither homoclinic nor heteroclinic orbits [12], and thus the Shilnikov theorem [13] cannot be used to verify the chaos.

Referring to *fractional-order* hyperchaotic systems *without equilibria*, to the best of our knowledge, no paper has been published in the literature so far. Based on this consideration, this study bridges the gap by introducing the first example of fractional hyperchaos. Specifically, the equations of the proposed system are:

$$\begin{cases} {}^{*}D^q x = y \\ {}^{*}D^q y = -x + yz + axzw \\ {}^{*}D^q z = 1 - y^2 \\ {}^{*}D^q w = z + bxz + cxyz \end{cases} \qquad (8)$$

where ${}^{*}D^q$ denotes the Caputo fractional operator defined in (2) with initial time $t_0 = 0$. It can be readily verified that the proposed system (8) has no equilibrium points. By applying the predictor-corrector algorithm described in Section 2, the solution of the fractional system (8) can be written as:

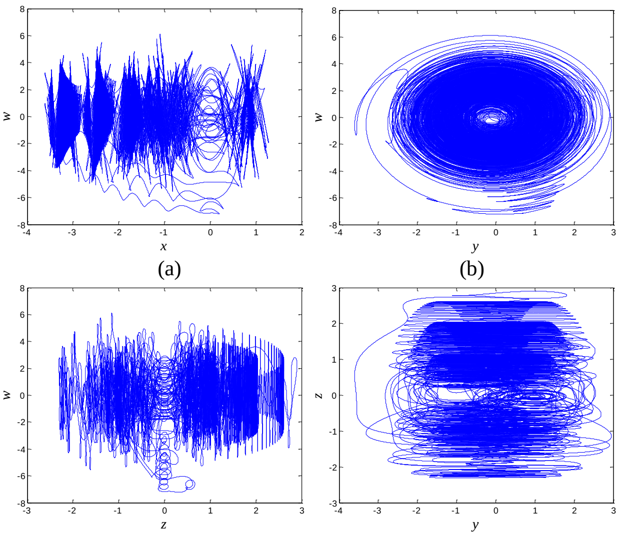
$$\begin{cases} x_h(t_{n+1}) = x_0 + \frac{h^q}{\Gamma(q+2)}\left(\hat{y}(t_{n+1})\right) + \frac{h^q}{\Gamma(q+2)} \sum_{j=0}^{n} \alpha_{j,n+1}\left(y(t_j)\right) \\ y_h(t_{n+1}) = y_0 + \frac{h^q}{\Gamma(q+2)}\left(-\hat{x}(t_{n+1}) + \hat{y}(t_{n+1})\hat{z}(t_{n+1}) + a\hat{x}(t_{n+1})\hat{z}(t_{n+1})\hat{w}(t_{n+1})\right) + \\ \qquad + \frac{h^q}{\Gamma(q+2)} \sum_{j=0}^{n} \alpha_{j,n+1}\left(-x(t_j) + y(t_j)z(t_j) + ax(t_j)z(t_j)w(t_j)\right) \\ z_h(t_{n+1}) = z_0 + \frac{h^q}{\Gamma(q+2)}\left(1 - \hat{y}(t_{n+1})\hat{y}(t_{n+1})\right) + \\ \qquad + \frac{h^q}{\Gamma(q+2)} \sum_{j=0}^{n} \alpha_{j,n+1}\left(1 - y(t_j)y(t_j)\right) \\ w_h(t_{n+1}) = w_0 + \frac{h^q}{\Gamma(q+2)}\left(\hat{z}(t_{n+1}) + b\hat{x}(t_{n+1})\hat{z}(t_{n+1}) + c\hat{x}(t_{n+1})\hat{y}(t_{n+1})\hat{z}(t_{n+1})\right) + \\ \qquad + \frac{h^q}{\Gamma(q+2)} \sum_{j=0}^{n} \alpha_{j,n+1}\left(z(t_j) + bx(t_j)z(t_j) + cx(t_j)y(t_j)z(t_j)\right) \end{cases} \qquad (9)$$

in which the predicted variables are:

$$\begin{cases} \hat{x}(t_{n+1}) = x_0 + \frac{1}{\Gamma(q)} \sum_{j=0}^{n} \beta_{j,n+1}\left(y(t_j)\right) \\ \hat{y}(t_{n+1}) = y_0 + \frac{1}{\Gamma(q)} \sum_{j=0}^{n} \beta_{j,n+1}\left(-x(t_j) + y(t_j)z(t_j) + ax(t_j)z(t_j)w(t_j)\right) \\ \hat{z}(t_{n+1}) = z_0 + \frac{1}{\Gamma(q)} \sum_{j=0}^{n} \beta_{j,n+1}\left(1 - y(t_j)y(t_j)\right) \\ \hat{w}(t_{n+1}) = w_0 + \frac{1}{\Gamma(q)} \sum_{j=0}^{n} \beta_{j,n+1}\left(z(t_j) + bx(t_j)z(t_j) + cx(t_j)y(t_j)z(t_j)\right) \end{cases}$$

$$\beta_{j,n+1} = \frac{h^q}{q}\left((n+1-j)^q - (n-j)^q\right), \qquad 0 \le j \le n. \qquad (10)$$

and $\alpha_{j,n+1}$ is given by (6). By considering the parameter values a = 8, b = −2.5 and c = −30, the discretized equations (9)-(10) are calculated for several values of order $0 < q < 1$. A remarkable finding of this paper is that hyperchaos exists in the proposed fractional system with no equilibria for the value q = 0.96. The phase portraits of the hyperchaotic attractor are shown in Fig.1.



(a) (b) (c) (d)

Fig. 1: Projections of the hyperchaotic attractor of the fractional system without equilibria (8) with q = 0.96: (a) (x,w)-plane, (b) (y,w)-plane, (c) (z,w)-plane, (d) (y,z)-plane.

The hyperchaotic nature of the attractor in Fig.1 is confirmed using the technique based on the recent paper [17]. Note that, while other numerical methods (like the Wolf algorithm) only give an estimation of the largest Lyapunov exponents, the algorithm in [17] is the only one able to provide the entire spectrum of Lyapunov exponents in fractional-order systems. The obtained spectrum (λ_1 = 0.91, λ_2 = 0.19, λ_3 = 0, λ_4 = -1.37) includes two positive values, confirming the hyperchaotic nature of the considered attractor.

4. Application to hyperchaos synchronization

Chaos synchronization between two dynamical systems (called drive and response system, respectively) consists in making the state variables of the response system synchronized in time with the drive system states [22]. A technique to obtain synchronization is the observer-based method, where the response system is designed to behave as an observer of the drive system [23]. Herein, an example of observer-based synchronization applied to the hyperchaotic fractional system (8) with q = 0.96 and a = 8, b = −2.5 and c = −30 is proposed. To this purpose, the drive system can be written in the form [22]-[23]:

$$\begin{pmatrix} {}^*D^q x(t) \\ {}^*D^q y(t) \\ {}^*D^q z(t) \\ {}^*D^q w(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \\ z(t) \\ w(t) \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y(t)z(t) + 8x(t)z(t)w(t) \\ -y^2(t) \\ -2.5x(t)z(t) - 30x(t)y(t)z(t) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \tag{11}$$

whereas the synchronizing vector signal $\boldsymbol{s}(t)$ is:

$$\boldsymbol{s}(t) = \begin{pmatrix} y(t)z(t) + 8x(t)z(t)w(t) \\ -y^2(t) \\ -2.5x(t)z(t) - 30x(t)y(t)z(t) \end{pmatrix} + \boldsymbol{K} \begin{pmatrix} x(t) \\ y(t) \\ z(t) \\ w(t) \end{pmatrix}, \tag{12}$$

where $\boldsymbol{K} \in \mathbf{R}^{3\times 4}$ is a gain matrix to be determined [23]. By applying the synchronization method proposed in [22-23], the response system is

$$\begin{pmatrix} {}^*D^q \hat{x}(t) \\ {}^*D^q \hat{y}(t) \\ {}^*D^q \hat{z}(t) \\ {}^*D^q \hat{w}(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \hat{x}(t) \\ \hat{y}(t) \\ \hat{z}(t) \\ \hat{w}(t) \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{y}(t)\hat{z}(t) + 8\hat{x}(t)\hat{z}(t)\hat{w}(t) \\ -\hat{y}^2(t) \\ -2.5\hat{x}(t)\hat{z}(t) - 30\hat{x}(t)\hat{y}(t)\hat{z}(t) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} (\boldsymbol{s}(t) - \hat{\boldsymbol{s}}(t)) \tag{13}$$

where $\hat{\boldsymbol{s}}(t)$ is the observer prediction of the synchronizing signal $\boldsymbol{s}(t)$, that is

$$\hat{\boldsymbol{s}}(t) = \begin{pmatrix} \hat{y}(t)\hat{z}(t) + 8\hat{x}(t)\hat{z}(t)\hat{w}(t) \\ -\hat{y}^2(t) \\ -2.5\hat{x}(t)\hat{z}(t) - 30\hat{x}(t)\hat{y}(t)\hat{z}(t) \end{pmatrix} + \boldsymbol{K} \begin{pmatrix} \hat{x}(t) \\ \hat{y}(t) \\ \hat{z}(t) \\ \hat{w}(t) \end{pmatrix}. \tag{14}$$

By defining the synchronization error between drive and response systems as

$$\begin{pmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \\ e_4(t) \end{pmatrix} = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \\ w(t) \end{pmatrix} - \begin{pmatrix} \hat{x}(t) \\ \hat{y}(t) \\ \hat{z}(t) \\ \hat{w}(t) \end{pmatrix}, \tag{15}$$

from equations (11)-(14) it can be shown that the following linear fractional-order error system is obtained:

$$\begin{pmatrix} {}^*D^q e_1(t) \\ {}^*D^q e_2(t) \\ {}^*D^q e_3(t) \\ {}^*D^q e_4(t) \end{pmatrix} = \left[\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \end{pmatrix} \right] \begin{pmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \\ e_4(t) \end{pmatrix} \tag{16}$$

It can be readily verified that the (4×12)-controllability matrix derived from (16) is full rank. Therefore, according to Theorem 1 stated in [24], the eigenvalues of the error system (16) can be assigned anywhere in the stability region defined by the following inequality:

$$\left| \arg\left(\operatorname{eig}\left[\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \\ k_{41} & k_{42} & k_{43} \end{pmatrix}^T \right] \right) \right| > 0.96\frac{\pi}{2} \tag{17}$$

Note that the complex region of asymptotic stability, defined by (17) for q = 0.96, is larger than the region corresponding to the integer-order case (the well-known open left half plane) since it includes a part of the right half plane shaped as a complementary wedge (Fig.2). This property of fractional systems can be exploited in the considered hyperchaotic synchronization. To this purpose, the eigenvalues are selected as $(0.2 \pm i9.5,\ 0.5 \pm i10.04)$, i.e. they have positive real parts but lie in the stability region depicted in Fig.2.

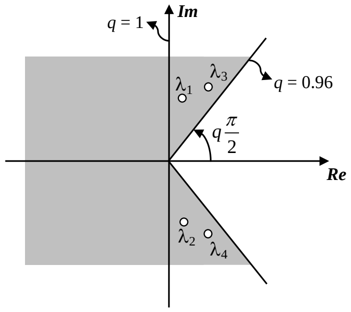


Fig. 2: The stability region of system (16) under condition (17) for q = 0.96 is in grey color.

Based on this choice, the following matrix is obtained

$$\begin{pmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \end{pmatrix} = \begin{pmatrix} 93.64 & -0.91 & -1 & 0.63 \\ -21.62 & -3.37 & -6.75 & 11.18 \\ 16.34 & -4.43 & -11.48 & 6.26 \end{pmatrix}, \tag{18}$$

which gives the linear error system

$$\begin{pmatrix} {}^*D^q_{t_0} e_1(t) \\ {}^*D^q_{t_0} e_2(t) \\ {}^*D^q_{t_0} e_3(t) \\ {}^*D^q_{t_0} e_4(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -94.64 & 0.91 & 1 & -0.63 \\ 21.62 & 3.37 & 6.75 & -11.18 \\ -16.34 & 4.43 & 12.48 & -6.26 \end{pmatrix} \begin{pmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \\ e_4(t) \end{pmatrix} \tag{19}$$

with eigenvalues given by $(0.2 \pm i9.5,\ 0.5 \pm i10.04)$. From Figure 3 it can be observed that the fractional error system

(19) is asymptotically stabilized at the origin (see inequality (17)), even though all the eigenvalues have positive real parts.

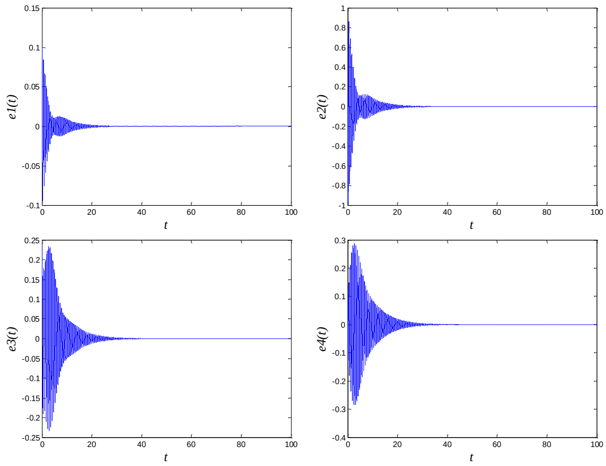


Fig. 3: Time behavior of the error system (19) for $q = 0.96$.

Note that the plots in Fig.3 are in agreement with the theoretical results expressed by the condition (17) and proved in [24]. This indicates that the observer-based method enables hyperchaotic synchronization between the fractional drive system (11) and the fractional response system (13) to be effectively achieved.

5. Conclusions

A new exciting phenomenon and unexplored field of research is represented by the presence of hyperchaos in fractional systems with no equilibria. This paper has investigated the topic by introducing a new example of fractional hyperchaotic system without equilibria. The approach has exploited the predictor-corrector algorithm to find the hyperchaotic attractor when the order of the derivative is q = 0.96. An application of the observer-based synchronization to the proposed hyperchaotic fractional system has been illustrated in detail.

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