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Analysis of a Switched Dynamical System with Spike Noise

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Abstract—The rigorous analysis and effect of spike noise are discussed in this paper. Firstly, we assume that spike noise arises in a system interrupted by own state and a periodic interval and explain its dynamics. The characteristics of a basic bifurcation phenomena which can be observed in a system with spike noise are examined by using the return map. Finally, we consider the dynamical effect of spike noise by using the two parameter bifurcation diagrams.

1. Introduction

The desecrate map of a continuous system interrupted by own state and a periodic interval constructs a piecewise smooth system [1]. In particular, the converter circuit is a typical example and a lot of studies have been done for the past few decades under the assumption of the theoretical switching action [2, 3, 4]. In contrast, Banerjee *et al* established that a capability of the missed switching and its greatly influences into the circuit dynamics based on the laboratory experiment [5]. This paper reported the mechanical response of a element or the floating capacitance and inductance causes the time delay or spike noise immediately after the switching action and exert a huge impact to the bifurcation structure.

On the other hand, most of previous researches hardly discussed switching non-ideality except Ref. [6]. Thus, we have proposed a simplest case of a system interrupted by own state and a periodic interval based on Ref. [5] and examined the fundamental mechanism of spike noise [7]. As the results, we found out that spike noise expands the invariant interval of a system and causes the parameter region of a coexisting attractors for the period-1 and period-2 solutions. However, the bifurcation analysis and investigation into the dynamical effect of spike noise in a widely parameter space are insufficient.

Therefore, we perform the rigorous bifurcation analysis in a system with spike noise and clarify the dynamical effects of it. The system for analysis is a simplest one [7] and we assume that spike noise arises with the switching action. We know that there are the period doubling bifurcation and border-collision bifurcation in a system with spike noise. The characteristics of each bifurcation phenomena are examined by using the return map. Finally, we derive the two parameter bifurcation diagrams and consider the dynamical effect of spike noise.

2. A switched dynamical system

We consider the following two differential systems.

$$\frac{dy}{d\tau} = -y + h(\tau, x),\ h(\tau, x) = \begin{cases} 1 & : \text{system A} \\ 0 & : \text{system B} \end{cases} \tag{1}$$

τ suggests a normalized time variable. Note that Eq. (1) can be relabeled as a following equations with the initial value y_k at $\tau = kT$.

$$y(\tau) = \begin{cases} (y_k - 1)e^{-(\tau - kT)} + 1 & : \text{system A} \\ y_k e^{-(\tau - kT)} & : \text{system B} \end{cases} \tag{2}$$

Figure 1 shows the dynamics of our system. Now, we assume that the waveform starts from y_k at $\tau = kT$ with the system A, and explain its behavior in a system with ideal switching (see Fig. 1 (a)). When the waveform reaches the reference value y_{r}, the system changes to B and a following clock pules bring it to A. Note that any clock pulse T before the intersection is ignored if the system is set at state A.

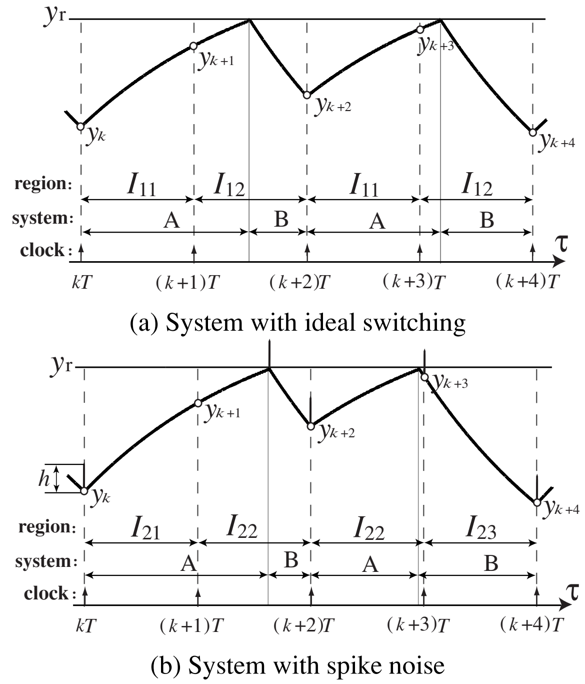
On the other hand, spike noise arises immediately after the every switching actions in a system with spike noise (see Fig. 1 (b)). The basic switching logic is same as a system with ideal switching. Suppose that when the reset pulse is impressed at every period of duration T, a system obeys state B if the peak of spike noise is higher than the reference value ($y_{\mathrm{r}} \leq y_k$). Also, spike noise has no influences if the peak of spike noise is lower than the reference value or a switching action at the reference value. It should be mentioned that we fix the size of spike noise as h in this paper based on Ref. [5].

3. Return map

To analyze a system in detail, we construct the return map. The waveform during the duration of clock interval T is classified by the border D_2 in a system with ideal switching.

$$D_2 = (y_{\mathrm{r}} - 1)e^T + 1 \tag{3}$$

So, we can observe the two types of waveform during the clock interval. The system obeys state A until the next clock pulse arrives in the first one. On the other hand, the



(a) System with ideal switching

(b) System with spike noise

Figure 1: Examples of the waveform.

system changes from state A to B during the clock interval in the other one. Now, we let the region which can be observed each types of the waveform as I_{11} and I_{12}

$$\begin{aligned} I_{11} &= \{y_{k-1}, y_k | y_k \leq D_2\} \\ I_{12} &= \{y_{k-1}, y_k | D_2 < y_k\} \end{aligned}, \tag{4}$$

Consequently, the return map of each region is given by

$$\begin{aligned} y_{k+1} &= G\,(y_k) \\ &= \begin{cases} (y_k - 1)\, e^{-T} + 1, & (y_{k-1}, y_k) \in I_{11} \\ y_{\mathrm{r}} \dfrac{y_k - 1}{y_{\mathrm{r}} - 1} e^{-T}, & (y_{k-1}, y_k) \in I_{12} \end{cases} \end{aligned} \tag{5}$$

In the following, we consider a system with spike noise. The waveform during the consecutive clock is classified into the three types by Eq. (3) and the border D_1.

$$D_1 = y_{\mathrm{r}} - h \tag{6}$$

Here, we assume that the border D_2 is smaller than the border D_1. On the same way, we let the region of a systemized waveform as I_{21}, I_{22} and I_{23} as follows.

$$\begin{aligned} I_{21} &= \{y_{k-1}, y_k | y_k \leq D_2\} \\ I_{22} &= \{y_{k-1}, y_k | (D_2 < y_k < D_1) \text{ or} \\ &\qquad\qquad (D_1 \leq y_k \text{ and } y_{k-1} < D_2)\} \\ I_{23} &= \{y_{k-1}, y_k | (D_1 \leq y_k) \text{ and } (D_2 \leq y_{k-1})\} \end{aligned} \tag{7}$$

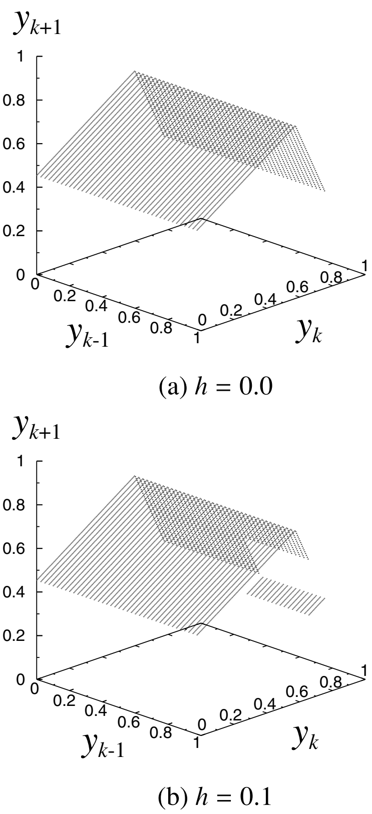
It should be aware that the system obeys state B during the duration T in the region I_{23}. This region is never seen in a system with ideal switching. Thus, the return map is defined as the following equation.

$$\begin{aligned} y_{k+1} &= F\,(y_k) \\ &= \begin{cases} (y_k - 1)\, e^{-T} + 1, & (y_{k-1}, y_k) \in I_{21} \\ y_{\mathrm{r}} \dfrac{y_k - 1}{y_{\mathrm{r}} - 1} e^{-T}, & (y_{k-1}, y_k) \in I_{22} \\ y_k e^{-T}, & (y_{k-1}, y_k) \in I_{23} \end{cases} \end{aligned} \tag{8}$$

Figure 2 shows the examples of return map on y_{k-1}-y_k-y_{k+1} plane. Suppose that a system with spike noise is classified into the two-dimensional model because a solution y_{k+1} depends on y_k and y_{k-1}. However the parameter y_{k-1} is only used in the classification of the map. Thus, we analyze our system by using the return map on y_k-y_{k+1} plane shown in Fig. 3, where this figure corresponds to Fig. 2.

4. Analytical results

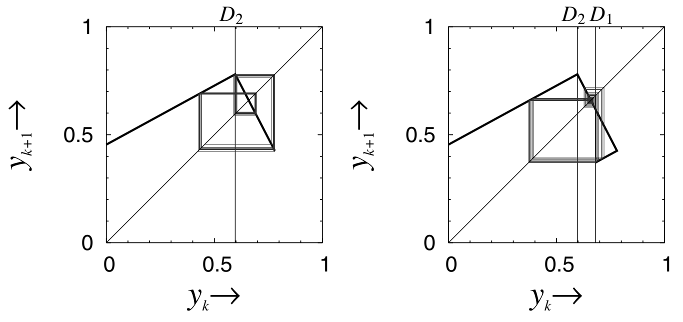
Figure 4 shows the two parameter bifurcation diagram in a system with ideal switching. In this figure, red and blue line means the period doubling bifurcation and border-collision bifurcation, respectively. Suppose that a detailed



(a) $h = 0.0$

(b) $h = 0.1$

Figure 2: Return map on y_{k-1}-y_k-y_{k+1} plane. ($T = 0.606$, $y_{\mathrm{r}} = 0.78$)



(a) $h = 0.0$ (b) $h = 0.1$

Figure 3: Return map on y_k-y_{k+1} plane. ($T = 0.606$, $y_r = 0.78$)

bifurcation analysis will be abbreviated in this paper because it has been already studied [1].

In a similar manner, the two parameter bifurcation diagrams in a system with spike noise are shown in Fig. 5. Where a solution which has positive Lyapunov exponent is regarded as a chaotic attractor in this paper. All of these solutions are included into the region of chaos in Figs. 4 and 5. For example, 0.14 and 0.25 are the solutions of Lyapunov exponent in Fig. 3 (a) (b). So these solutions are classified into the chaotic attractor. Note that, a gray region shows a parameter space of $D_1 < D_2$. This region is not practical in terms of a switching logic of our system. The period doubling bifurcation and border-collision bifurcation can be also observed in a system with spike noise. In the following, we explain the fundamental characteristics of the each bifurcation phenomena.

In our system, the period doubling bifurcation occurs and system dynamics becomes unstable. After that a part of the solution intersects border D_2, and subharmonic or chaotic solutions are generated. We know that the period-m waveform which is concerned to the period doubling bifurcation has $m-1$ mapping points in I_{21} and 1 mapping point in I_{22}. Note that the period-m waveform has m mapping points in Eq. (8). Furthermore, it is well known that the period doubling bifurcation occurs if the solution of a characteristic equation μ satisfies $\mu = -1$. Thus, we can define the specific condition of the period doubling bifurcation as

$$y_r\left(1 + e^{-mT}\right) - 1 = 0, \tag{9}$$

where Eq. (9) corresponds to the red line in Fig. 5.

The border-collision bifurcation occurs when the solution intersects border D_1. After that the waveform changes another one. In our system, the period-m waveform which is concerned to this bifurcation phenomena can be classified into two types as follows.

Type 1: The map has $m-2$ points in I_{21} and 1 point in I_{22} and I_{23}.

Type 2: There are 1 mapping point in I_{22} and $m-1$ mapping points in I_{23}.

Both of above two types of m-periodic waveforms preface border-collision bifurcation. In terms of that, the border-collision bifurcation in a system with spike noise can be classified into two types. Suppose that we call each types of the border-collision bifurcation as the border-collision bifurcation-1 and border-collision bifurcation-2, respectively. We also find out that the period-m waveform changes only to the period-$m+1$ waveform or vice versa through the border-collision bifurcation-2. Where the specific condition of the border-collision bifurcation-1 can be calculated as the following equation.

$$\begin{aligned} y_r &= f_1(m, T, h) \\ &= \Big\{ -\left(h + 1 - \left(h + e^T\right) e^{-mT}\right) \\ &\quad + \sqrt{\{h + 1 - (h + e^T)\, e^{-mT}\}^2 - 4h\,\{e^{-mT} - 1\}} \Big\} \\ &\quad \Big/ \left\{2\left(e^{-mT} - 1\right)\right\} \end{aligned} \tag{10}$$

Likewise, the condition of border-collision bifurcation-2 can be defined as

$$\begin{aligned} y_r &= f_2(m, T, h) \\ &= \frac{1}{2}\left\{(h + 1) - \sqrt{(h + 1)^2 - \frac{4h}{1 - e^{-mT}}}\right\}. \end{aligned} \tag{11}$$

Where Eqs. (10) and (11) are correspond to the green and yellow lines in Fig. 5.

In the following, we discuss the dynamical effect of spike noise by comparing the two parameter bifurcation diagrams. First of all, we pay attention to the expansion of a existence region of the period-m solutions in the system with spike noise. Where we consider the stability of a periodic solutions. In a system with ideal switching, the periodic solution bifurcates into the chaotic attractor when

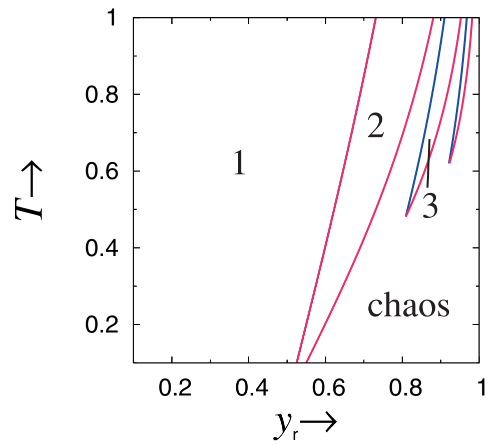
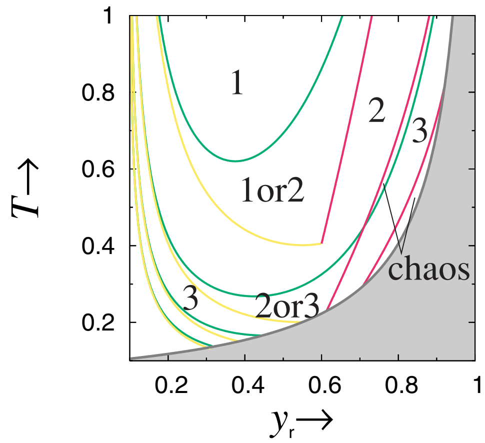


Figure 4: Two parameter bifurcation diagram in the system with ideal switching. ($h = 0.0$)

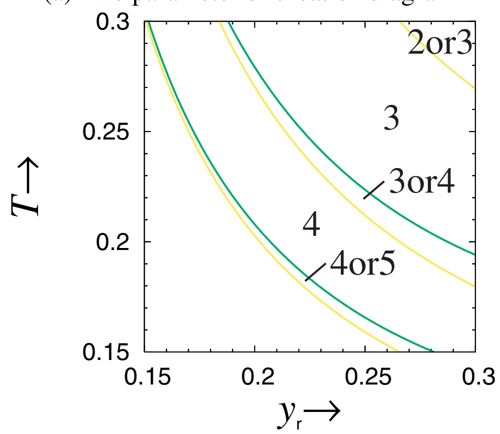
the mapping point intersects border D_2, however, it can be kept until the mapping point intersects border D_1 in a system with spike noise. In other words, the new region that the system obeys state B during the clock interval causes the new types of border-collision bifurcation in the system with spike noise, and it has huge impact to the existence region of the periodic solutions. Then, we focus on the parameter region of the coexisting attractors. We have already clarified that the period-1 and period-2 waveform coexist at the same parameter [7]. In addition, we find out various kinds of coexisting attractors in a system with spike noise (see Fig. 5 (b)). These coexisting regions can be defined as the following equation.

$$f_2(m, T, h) < y_r < f_1(m+1, T, h) \tag{12}$$

Suppose that spike noise has no effect to the period doubling bifurcation because of a solution of characteristics equation can be derived same form between the system with ideal switching and spike noise. On the other hand, we find out I_{23} makes the new types of border-collision bifurcations and they greatly influences to the system dynamics.



(a) Two parameter bifurcation diagram



(b) Enlarged illustration of (a)

Figure 5: Two parameter bifurcation diagrams. ($h = 0.1$)

5. Conclusion

This paper presented a rigorous analysis and dynamical effect of spike noise in a system interrupted by own state and a periodic interval. Suppose that our system is a simplest special case constructed by the switching part of converter circuit [5], and we artificially added spike noise on it. The fundamental bifurcation phenomena were examined by using the return map. Finally, we derived some two parameter bifurcation diagrams and considered dynamical effect of spike noise. As the results, we found out that the border-collision bifurcation greatly influences to the system dynamics. Moreover, we discovered a parameter region of the coexisting attractors of the period-m and period-$m+1$ solutions. The analytical results of this paper may apply to another switching systems which have spike noise too. Our future work to be studied is analysis of more high-dimensional systems with missed switching.

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