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Stability of a Switched System for Continuous-time Tomographic Image Reconstruction

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Abstract—We have presented a novel approach for reconstructing tomographic images based on the idea of continuous dynamical methods. The method consists of a continuous-time image reconstruction (CIR) system described by differential equations for solving ill-posed inverse problems. We see that a switched system with a piecewise smooth vector field, which describes a block CIR system, can reconstruct better images than the smooth CIR system. For investigating an essential property of the switched system, we considered a small system and gave its analytic solution as well as sufficient conditions for its stability via multiple Lyapunov functions approach. However, some results seem to be incorrect. In this paper, we reprove stability, but via the common Lyapunov function approach which requires the existence of a single Lyapunov function whose derivative along solutions of all subsystems satisfies suitable inequalities.

1. Introduction

Tomography is imaging by sections or sectioning, and computed tomography(CT) [1, 2, 3, 4] is a medical imaging method employing tomography created by computer processing. In general, a problem of reconstructed images from a projection operator and a projection data set become ill-posed. Many different reconstruction algorithms are used in medical practice to solve the inverse problem of image reconstruction and most algorithms fall into one of two categories, filtered back projection (FBP) which is a transform method and iterative methods using difference equations that we call them discrete-time image reconstraction(DIR); FBP demands fewer computational resources, while DIR generally produces fewer artifacts at a higher computing cost; moreover, there are continuous dynamical methods that can regularize such ill-posed inverse problems.

We have presented a novel approach [5] for reconstructing tomographic images based on the idea of continuous dynamical methods [6, 7, 8, 9, 10]. The method consists of a continuous-time image reconstruction (CIR) system [5, 11] described by differential equations for solving ill-posed [12, 13, 14] inverse problems. We see that a switched system with a piecewise smooth vector field, which describes a block CIR system, can reconstruct better images than the smooth CIR system [15]. For investigating an essential property of the switched system, we considered a small system and gave the analytic solution of it, which is positive when we start with a positive initial value coinciding with what we already have proved in the general case [15], as well as sufficient conditions for its stability via multiple Lyapunov functions approach [16]. However, proving stability by using multiple Lyapunove functions approach [16] is incorrect. In this paper, in the light of the standard Lyapunov theory for smooth systems [17, 18], we reprove stability, but via the common Lyapunov function approach [19] which requires the existence of a single Lyapunove function whose derivative along solutions of all subsystems satisfies suitable inequalities; to do that, we investigate an expression for the candidate function and its derivatives with respect to all subsystems and then go back to choose the parameters of that function so as to make the candidate function positive definite and its derivatives negative definite for all subsystems.

2. Block CIR System [5, 11, 15]

The basic problem of computed tomography (CT) is to calculate the pixel values $\boldsymbol{x} \in \mathbb{R}^J_+$, with $\mathbb{R}_+$ denoting the set of non-negative real numbers, satisfying

$$\boldsymbol{y} = A\boldsymbol{x}, \tag{1}$$

where $\boldsymbol{y} \in \mathbb{R}^I_+ \backslash \{\boldsymbol{0}\}$ is the projection value, and $A \in \mathbb{R}^{I\times J}_+ \backslash \{\boldsymbol{0}\}$ is a normalized projection operator. For inconsistent projection data, Eq. (1) is an ill-posed problem, which means that its solution is not unique or does not exist.

In our previous papers, to find a solution $\boldsymbol{x}$, we formulated an optimization problem described as:

$$\min_{\boldsymbol{x}(t)\in\mathbb{R}^J_+} V(\boldsymbol{x}(t)), \quad t \in \mathbb{R},$$
$$V(\boldsymbol{x}) := \frac{1}{2}\|\boldsymbol{y} - A\boldsymbol{x}\|^2_2. \tag{2}$$

and to obtain a local minimum of the objective function, we proposed a continuous dynamical method as an initial value problem in the following form:

$$\begin{aligned} \frac{d\boldsymbol{x}}{dt} &= -\boldsymbol{X}\frac{\partial V(\boldsymbol{x})}{\partial \boldsymbol{x}}^{\top} \\ &= \boldsymbol{X}\boldsymbol{A}^{\top}(\boldsymbol{y}-\boldsymbol{A}\boldsymbol{x}), \\ & \quad t \in \mathbb{R}_{+}, \quad \boldsymbol{x}(0)=\boldsymbol{x}_0, \end{aligned} \tag{3}$$

where $\boldsymbol{X}$ indicates the diagonal matrix of order $J \times J$ in which the corresponding diagonal elements are elements of $\boldsymbol{x}$. The nonlinear system (3) has the property that $V(\boldsymbol{x})$, which can be a Lyapunov function on the state space $\mathbb{R}^{J}_{++}$, decreases along the solution $\boldsymbol{\phi}(t, \boldsymbol{x}_0)$ in time through the initial state $\boldsymbol{x}_0 \in \mathbb{R}^{J}_{++}$, with $\mathbb{R}_{++}$ representing the set of positive real numbers.

We also proposed a block CIR system by introducing subsets of projections as in block-iterative DIR methods. Let $\boldsymbol{B}_m \in \mathbb{R}^{I_m \times J}_{+}$ and $z_m \in \mathbb{R}^{I_m}_{+}$ be, respectively, a submatrix consisting of I_m partial rows of $\boldsymbol{A}$ and a subvector of $\boldsymbol{y}$ with the same corresponding rows of $\boldsymbol{B}_m$, for $m = 1, 2, \ldots, M$, such that there exists an elementary matrix $\boldsymbol{Q}$ satisfying:

$$\boldsymbol{Q}\begin{pmatrix} \boldsymbol{B}_1 \\ \boldsymbol{B}_2 \\ \vdots \\ \boldsymbol{B}_M \end{pmatrix} = \boldsymbol{A} \text{ and } \boldsymbol{Q}\begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_M \end{pmatrix} = \boldsymbol{y}. \tag{4}$$

The block CIR system was defined by

$$\begin{aligned} \frac{d\boldsymbol{x}}{dt} &= \boldsymbol{X}\boldsymbol{B}_m^{\top}(z_m - \boldsymbol{B}_m\boldsymbol{x}), \\ & \quad t - k\tau \in [t_{m-1}, t_m), \quad t \in \mathbb{R}_{+}, \\ & \quad \boldsymbol{x}(0) = \boldsymbol{x}^0 \in \mathbb{R}^{J}_{++}, \end{aligned} \tag{5}$$

for a series of times $0 = t_0 < t_1 < t_2 < \ldots < t_M = \tau$ and a non-negative integer k. This is a periodic non-autonomous system when $M \geq 2$ and is an autonomous system described by Eq. (3) when $M = 1$. The subsystem of Eq. (5) with $M \geq 2$ defined in the time interval $t - k\tau \in [t_{m-1}, t_m)$ for any m and k is described by:

$$\frac{d\boldsymbol{x}}{dt} = -\boldsymbol{X}\frac{\partial V_m(\boldsymbol{x})}{\partial \boldsymbol{x}}^{\top}, \tag{6}$$

where,

$$V_m(\boldsymbol{x}) = \frac{1}{2}\|z_m - \boldsymbol{B}_m\boldsymbol{x}\|_2^2. \tag{7}$$

Each subsystem of Eq. (5) has the property that $V_m(\boldsymbol{x})$, which can be a Lyapunov function, decreases in time along the solution starting from $\boldsymbol{x}(k\tau + t_{m-1}) \in \mathbb{R}^{J}_{++}$.

We theoretically, numerically demonstrated that our CIR, Eq. (3), and block CIR, Eq. (5), systems does not produce unphysically negative pixel values for positive initial values; and also, by using examples, we indicated that the quality of the reconstracted images by our systems was better than that from other methods.

3. Switched Systems [19, 20]

Suppose that we are given a family $\boldsymbol{f}_p$, $p \in P$ of functions from $\mathbb{R}^n$ to $\mathbb{R}^n$, where P is some index set. This gives rise to a family of systems:

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{f}_p(\boldsymbol{x}), \quad p \in P \tag{8}$$

evolving on $\mathbb{R}^n$. The functions $\boldsymbol{f}_p$ are assumed to be sufficiently regular. The easiest case to think about is when all these systems are linear and the index set P is finite: $P = \{1, 2, \ldots, m\}$. The switched system with time-dependent switching, generated by the above family, can be described by the equation:

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{f}_{\sigma}(\boldsymbol{x}), \tag{9}$$

where the switching signal $\sigma : [0, \infty) \to P$ is a piecewise constant function which has a finite number of discontinuities, which we call the switching times, on every bounded time interval and takes a constant value on every interval between two cosecutive switching times. The role of σ is to specify, at each time instant t, the index $\sigma(t) \in P$ of the active subsystem, i.e., the system from the family (8) that is currently being follwed.

It is well known that a necessary condition for asymptotic stability under arbitrary switching is that all of the individual subsystems are asymptotically stable. In fact, this condition is not sufficient for asymptotic stability under arbitrary switching. The following theorem gives additional requirements, on the systems from the family (8), that guarantee asymptotic stability of the switched system (9) for arbitrary switching signals.

Definition 1. *Consider a function $V : \mathbb{R}^n \to \mathbb{R}$. It is called positive definite if $V(\mathbf{0}) = 0$ and $V(\boldsymbol{x}) > 0$ for all $\boldsymbol{x} \neq \mathbf{0}$. If $V(\boldsymbol{x}) \to \infty$ as $|\boldsymbol{x}| \to \infty$, V is said to be radially unbounded.*

Definition 2. *Given a positive definite continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}$, we will say that it is a common Lyapunov function for the family of systems (8) if there exists a positive definite continuous function $W : \mathbb{R}^n \to \mathbb{R}$ such that we have*

$$L_{f_p}V(\boldsymbol{x}) = \frac{\partial V}{\partial \boldsymbol{x}}\boldsymbol{f}_p(\boldsymbol{x}) \leq -W(\boldsymbol{x}) \quad \forall \boldsymbol{x}, \quad \forall p \in P. \tag{10}$$

Theorem 1. *If all systems in the family (8) share a radially unbounded common Lyapunov function, then the switched system (9) is globally uniformly asymptotically stable.*

The terminology uniform is employed here to indicate the uniformity with respect to the switching signals, while the term global refers to the fact that it holds for all initial states.

4. Main Result

We studied a block CIR system (5) with $I = J = M = 2$, which is defined for $t \in \left[k\tau, (k + \frac{1}{2})\tau\right)$ by

$$\frac{dx_1}{dt} = x_1(y_1 - x_1), \tag{11}$$

$$\frac{dx_2}{dt} = 0, \tag{12}$$

and for $t \in \left[(k + \frac{1}{2})\tau, (k + 1)\tau\right)$ by

$$\frac{dx_1}{dt} = x_1\left(y_2 - (x_1 + x_2)\right), \tag{13}$$

$$\frac{dx_2}{dt} = x_2\left(y_2 - (x_1 + x_2)\right), \tag{14}$$

where k is a non-negative integer, and got its analytic solution which is positive when we start with a positive initial state [16].

Now, we can apply Theorem 1 by looking for a common Lyapunov function $V(\boldsymbol{x})$ that would have satisfy Eq. (10). The two subsystems have the point $(y_1, y_2 - y_1)$ as a common equilibrium point which is stable for each of the two subsystems by virtue of existing two Lyapunov functions given by Eq. (7). Consider the following quadratic form:

$$\begin{aligned} V(\boldsymbol{x}) &= \frac{1}{2}\begin{pmatrix} y_1 - x_1 \\ y_2 - x_1 - x_2 \end{pmatrix}^{\top}\begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix}\begin{pmatrix} y_1 - x_1 \\ y_2 - x_1 - x_2 \end{pmatrix} \\ &= \frac{1}{2}p_{11}(y_1 - x_1)^2 + p_{12}(y_1 - x_1)(y_2 - x_1 - x_2) \\ &\quad + \frac{1}{2}p_{22}(y_2 - x_1 - x_2)^2, \end{aligned} \tag{15}$$

for some positive definite matrix

$$P = \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix}$$

whose elements are functions of x_1 and x_2. For the quadratic form to be positive definite, the elements of the matrix P must satisfy

$$p_{11} > 0, \quad p_{11}p_{22} - p_{12}^2 > 0. \tag{16}$$

The derivative $\dot{V}(\boldsymbol{x})$ is given by

$$\begin{aligned} \dot{V}(\boldsymbol{x}) &= p_{12}\left[-(y_1 - x_1)(\dot{x}_1 + \dot{x}_2) - (y_2 - x_1 - x_2)\dot{x}_1\right] \\ &\quad - p_{11}(y_1 - x_1)\dot{x}_1 - p_{22}(y_2 - x_1 - x_2)(\dot{x}_1 + \dot{x}_2) \\ &\quad + \dot{p}_{12}(y_1 - x_1)(y_2 - x_1 - x_2) \\ &\quad + \frac{1}{2}\dot{p}_{11}(y_1 - x_1)^2 + \frac{1}{2}\dot{p}_{22}(y_2 - x_1 - x_2)^2. \end{aligned} \tag{17}$$

Substituting from Eq. (11) and Eq. (12) into Eq. (17), we get

$$\begin{aligned} L_{f_1}V(\boldsymbol{x}) &= -x_1(p_{11} + p_{12})(y_1 - x_1)^2 \\ &\quad - x_1(p_{12} + p_{22})(y_1 - x_1)(y_2 - x_1 - x_2) \\ &\quad + \frac{1}{2}\dot{p}_{11}(y_1 - x_1)^2 + \dot{p}_{12}(y_1 - x_1)(y_2 - x_1 - x_2) \\ &\quad + \frac{1}{2}\dot{p}_{22}(y_2 - x_1 - x_2)^2. \end{aligned} \tag{18}$$

Substituting from Eq. (13) and Eq. (14) into Eq. (17), we get

$$\begin{aligned} L_{f_2}V(\boldsymbol{x}) &= -\left[(x_1 + x_2)p_{22} + x_1 p_{12}\right](y_2 - x_1 - x_2)^2 \\ &\quad - (x_1 p_{11} + x_1 p_{12} + x_2 p_{12})(y_1 - x_1)(y_2 - x_1 - x_2) \\ &\quad + \frac{1}{2}\dot{p}_{11}(y_1 - x_1)^2 + \dot{p}_{12}(y_1 - x_1)(y_2 - x_1 - x_2) \\ &\quad + \frac{1}{2}\dot{p}_{22}(y_2 - x_1 - x_2)^2. \end{aligned} \tag{19}$$

Now we want to choose p_{11}, p_{12}, and p_{22} such that $L_{f_1}V(\boldsymbol{x})$ and $L_{f_2}V(\boldsymbol{x})$ are negative definite. Since the cross product $(y_1 - x_1)(y_2 - x_1 - x_2)$ is sign indefinite, we will cancel it by taking

$$p_{22} = -p_{12}, \quad p_{11} = -\left(\frac{x_1 + x_2}{x_1}\right)p_{12}. \tag{20}$$

With these choices, p_{12} must be negative for $V(\boldsymbol{x})$ to be positive definite via the satisfaction of Eq. (16). To simplify our choices, let us take p_{12} as a constant and so, by using Eq. (20), we get

$$\dot{p}_{22} = \dot{p}_{12} = 0, \quad \dot{p}_{11} = -\left(\frac{x_1\dot{x}_2 - x_2\dot{x}_1}{x_1^2}\right)p_{12}, \tag{21}$$

which implies that

$$\dot{p}_{11} = \frac{x_2}{x_1}(y_1 - x_1)p_{12}, \quad \dot{p}_{11} = 0, \tag{22}$$

for the first and second subsystems, respectively. Using the above discussion to get $L_{f_1}V(\boldsymbol{x})$ and $L_{f_2}V(\boldsymbol{x})$ as

$$L_{f_1}V(\boldsymbol{x}) = \frac{1}{2}x_2\left(1 + \frac{y_1}{x_1}\right)(y_1 - x_1)^2 p_{12}, \quad p_{12} < 0, \tag{23}$$

and

$$L_{f_2}V(\boldsymbol{x}) = x_2(y_2 - x_1 - x_2)^2 p_{12}, \quad p_{12} < 0. \tag{24}$$

Hence, the common Lyapunov function takes the form

$$\begin{aligned} V(\boldsymbol{x}) &= -\frac{1}{2}\left(\frac{x_1 + x_2}{x_1}\right)(y_1 - x_1)^2 p_{12} - \frac{1}{2}(y_2 - x_1 - x_2)^2 p_{12} \\ &\quad + (y_1 - x_1)(y_2 - x_1 - x_2)p_{12} \quad p_{12} < 0. \end{aligned} \tag{25}$$

If we take $p_{12} = -1$, for example, we will get the common Lyapunov function

$$\begin{aligned} V(\boldsymbol{x}) &= \frac{1}{2}\left(\frac{x_1 + x_2}{x_1}\right)(y_1 - x_1)^2 + \frac{1}{2}(y_2 - x_1 - x_2)^2 \\ &\quad - (y_1 - x_1)(y_2 - x_1 - x_2). \end{aligned} \tag{26}$$

Now we introduce two figures, the first one, Figure 1, shows the value of the common Lyapunov function given by Eq. (26), while the second one, Figure 2, shows a trajectory approaches the common equilibrium point $(y_1, y_2 - y_1)$.

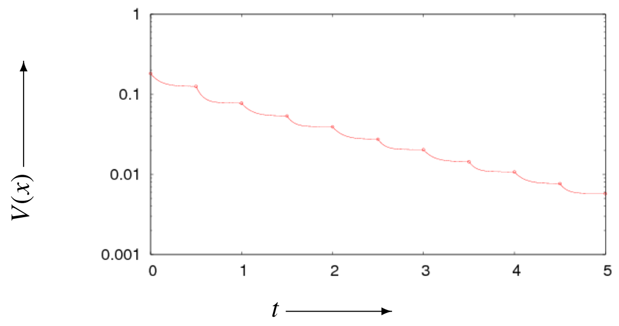

Figure 1: Common Lyapunov Function

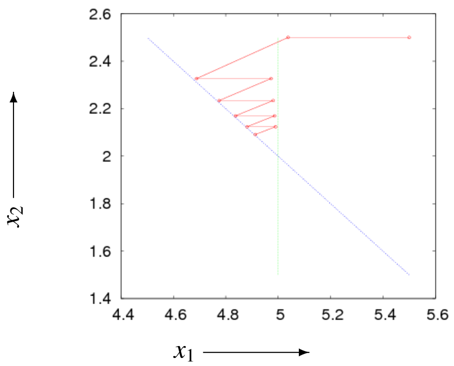

Figure 2: Trajectory of Switched System

5. Conclusion

In this study, for investigating an essential property of our new approach, Block CIR system, for reconstructing tomographic images, we considered a small switched system and discussed its stability via the common Lyapunov function approach which shows stability for any switching signal.

References

[1] H. Stark, Image recovery: Theory and applications, Academic, Florida, 1987.

[2] A.C. Kak, and M. Slaney, Principles of computerized tomographic imaging, IEEE Press, Piscataway, NJ, 1987.

[3] R. Gordon, R. Bender, and G.T. Herman, Algebraic reconstruction techniques (ART) for three-dimensional electron microscopy and X-ray photography, J. Theor. Biol., vol.29, no.3, pp.471–481, 1970.

[4] L.A. Shepp, and Y. Vardi, Maximum likelihood reconstruction for emission tomography, IEEE Trans. Med. Imag. vol.1, no.2, pp.113–122, 1982.

[5] K. Fujimoto, O.M. Abou Al-Ola and T. Yoshinaga, Continuous-Time Image Reconstruction Using Differential Equations for Computed Tomography, Communications in Nonlinear Science and Numerical Simulations, vol.15, no.6, pp.1648–1654, 2010.

[6] J. Schropp, Using dynamical systems methods to solve minimization problems, Appl. Numer. Math., vol.18, no.1, pp.321–335, 1995.

[7] R.G. Airapetyan, A.G. Ramm, and A.B. Smirnova, Continuous analog of gauss-newton method, Math. Models Meth. Appl. Sci., vol.9, no.3, pp.463–474, 1999.

[8] A.G. Ramm, Linear ill-posed problems and dynamical systems, J. Math. Anal. Appl., vol.258, no.1, pp.448–456, 2001.

[9] A.G. Ramm, Dynamical systems method for solving operator equations, Commun. Nonlin. Sci. Numer. Simul., vol.9, no.4, pp.383–402, 2004.

[10] L. Li, and B. Han, A dynamical system method for solving nonlinear ill-posed problems, Appl. Math. Comput., vol.197, no.1, pp.399–406, 2008.

[11] O.M. Abou Al-Ola, K. Fujimoto and T. Yoshinaga, Dynamics of Continuous-Time Image Reconstruction System for Computed Tomography, Proc. of 2009 International Symposium on Nonlinear Theory and its Applications, pp.627–630, 2009.

[12] J. Hadamard, Sur les problèmes aux dérivées partielles et leur signification physique, Princeton university bulletin, Princeton University, 1902.

[13] H.W. Engl, K. Kunisch, and A. Neubauer, Convergence rates for Tikhonov regularization of nonlinear ill-posed problems, Inverse Probl., vol.5, pp.523–540, 1989.

[14] M. Hanke, A. Neubauer, and O. Scherzer, A convergence analysis of the landweber iteration for nonlinear ill-posed problems, Numer. Math., vol.72, no.1, pp.21–37, 1995.

[15] O.M. Abou Al-Ola, K. Fujimoto and T. Yoshinaga, Property of Equilibrium Existed in Continuous-Time Image Reconstruction System With Subsets for Medical Tomographic Image, Proc. of the Electronics, Information and Systems Conference Electronics, Information and Systems Society, I.E.E. of Japan, pp.951–956, 2009.

[16] O.M. Abou Al-Ola, K. Fujimoto and T. Yoshinaga, Properties of a switched system for continuous-time tomographic image reconstruction, Proc. of the 2010 RISP International Workshop on Nonlinear Circuits and Signal Processing, pp.219–222, 2010.

[17] H.K. Khalil, Nonlinear Systems, 3rd edition, Prentice Hall, New Jersy, 2002.

[18] A.M. Lyapunov, Stability of motion, Acad. Press 1966.

[19] D. Liberzon, Switching in Systems and Control, Birkhauser, Boston, 2003.

[20] M.S. Branicky, Multiple Lyapunov Functions and Other Analysis Tools for Switched and Hybrid Systems, IEEE Transactions on Automatic Control, vol.43, no.4, pp.475–482, 1998.