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Transition to chaos through wrinkling of mode locked torus in a piecewise smooth map

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Abstract—We demonstrate a mechanism of breakdown of a two-frequency torus in a piecewise smooth map, resulting in the transition from a mode-locked periodic orbit to chaos. It occurs through wrinkling of the nonsmooth unstable manifold that makes its length extend to infinity. We explain the mechanism using the concept of stable foliation.

1. Introduction

There are many routes for the transition from a periodic orbit to a chaotic orbit, the most well-known ones being the period-doubling route and the quasiperiodicity route. In the latter route, a two-frequency torus forms through a Neimark-Sacker bifurcation, which is subsequently destroyed to give rise to a chaotic orbit.

The basic theorem for the destruction of two frequency torus was given by Afraimovich and Shilnikov [1], where three possible routes were described. References [2, 3] give an overview of the possible topological transitions for the loss of smoothness of the torus. Later these routes have been confirmed numerically as well as experimentally for both continuous- and discrete-time systems [4, 5]. These results were obtained in the context of smooth maps.

In recent years, piecewise smooth maps (PWS) have attracted significant research attention because such maps have been found to model many physical, engineering, biological systems [6]. In a series of recent publications, the creation of a torus has been described [8] for piecewise smooth systems. It has also been shown [9] that there can be transitions between resonance torus and ergodic torus as a parameter is varied, and the mechanism of the transition involves a breakdown of the torus through homoclinic intersection. There is a 'first contact' between the stable and the unstable manifold, followed by the homoclinic intersection, and then there is a 'second contact' after which the homoclinic structure ceases to exist. It was claimed that after the second homoclinic contact the behavior changes to quasiperiodicity.

Our further investigation has revealed that the orbit generated after the second homoclinic contact is, in fact, chaotic—a large circle-like strange attractor. In this paper we explain the mechanism of occurrence of such a strange attractor.

In our present investigation we follow the bifurcations that take place within a 1 : 5 mode locking tongue. In particular, we consider the same parameter values as used in [9]. We show that the resonance torus is first destroyed through homoclinic bifurcation and then quadratic tangencies with the stable foliation after the last homoclinc tangency is responsible for the occurrence of the strange attractor.

2. The Normal Form Map and the Formation of Resonant Torus

We consider such PWS maps whose leading order Taylor term in the neighborhood of the border is linear. For such maps the normal form can be expressed as [7]

$$\begin{pmatrix} x^{(n+1)} \\ y^{(n+1)} \end{pmatrix} = \begin{cases} \begin{pmatrix} \tau_L & 1 \\ -\delta_L & 0 \end{pmatrix}\begin{pmatrix} x^{(n)} \\ y^{(n)} \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}\mu, & \text{if } x^{(n)} < 0 \\ \begin{pmatrix} \tau_R & 1 \\ -\delta_R & 0 \end{pmatrix}\begin{pmatrix} x^{(n)} \\ y^{(n)} \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}\mu, & \text{if } x^{(n)} > 0 \end{cases} \tag{1}$$

The phase space is divided into two regions $L = \{(x, y) : x \leq 0, y \in \mathbb{R}\}$ and $R = \{(x, y) : x > 0, y \in \mathbb{R}\}$. The fixed points of the system (1) in both sides of the boundary are given by:

$$L^* = \left(\frac{\mu}{1 + \delta_L - \tau_L}, \frac{-\delta_L \mu}{1 + \delta_L - \tau_L}\right), \quad R^* = \left(\frac{\mu}{1 + \delta_R - \tau_R}, \frac{-\delta_R \mu}{1 + \delta_R - \tau_R}\right) \tag{2}$$

If the x-component of L^* is negative, the fixed point exists and we call it a real fixed point. However, when the x-component of L^* is positive, we call it a virtual one and then it is denoted by $\bar{L}^*$. Similarly, when the x-component of R^* is positive, the fixed point exists; else it is a virtual fixed point denoted by $\bar{R}^*$.

The stability of L^* and R^* are determined by the eigenvalues

$$\lambda_{L\pm} = \frac{1}{2}\left(\tau_L \pm \sqrt{\tau_L^2 - 4\delta_L}\right), \quad \lambda_{R\pm} = \frac{1}{2}\left(\tau_R \pm \sqrt{\tau_R^2 - 4\delta_R}\right).$$

The theory of border collision bifurcations developed so far mainly assumes that $|\delta_L| < 1$ and $|\delta_R| < 1$, so that the system is dissipative. Recently some researchers have shown [9, 10] that many interesting dynamics can occur when on one side of the border the determinant of the jacobian matrix is greater than one. Accordingly, we choose the parameters in the range $|\delta_L| < 1$ and $|\delta_R| > 1$. The conditions

$$-(1+\delta_L) < \tau_L < (1+\delta_L), \quad -2\sqrt{\delta_R} < \tau_R < 2\sqrt{\delta_R}$$

then ensure that the fixed point is attracting for $\mu < 0$ and is a spiral repeller for $\mu > 0$. In our present investigation we have used the values $\delta_L = 0.5$ and $\delta_R = 1.6$.

As we vary the parameter μ from a negative value to a positive value, the fixed point moves from L to R through border collision. With the traces and determinants set within the above range, at the bifurcation point $\mu = 0$, the fixed point turns into an unstable focus. If we start from any point on the R side, the iterates will start to spiral outward and as it crosses the boundary and comes on the L side, the systems dynamics will be governed by the virtual attractor $\bar{L}^*$ situated in R. As a result, the outward motion of the system is arrested and it displays a rotating motion governed by the complex conjugate eigenvalues of the unstable focus situated on the R side.

We have shown the two-parameter (τ_L, τ_R) bifurcation diagram in Fig. 1, obtained for $\mu > 0$. This diagram mainly consists of various periodic tongues. These resonance tongues exhibit a lens-chain structure. This was first reported in [11] for a piecewise linear circle map and later it has been observed for the normal form map (1) in [9].

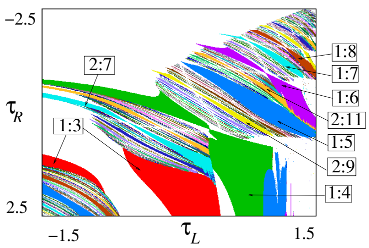

Figure 1: Two-parameter bifurcation diagram of the normal form map in the parameter plane (τ_L, τ_R) with $\delta_L = 0.5$, $\delta_R = 1.6$, and $\mu = 0.05$.

3. Large Circle-Like Strange Attractor

If we choose the parameter values corresponding to a tongue of periodicity, the stable and unstable cycles arise from the fixed point through border collision bifurcation. Depending on the parameter space these cycles along with the unstable manifold of the saddle cycle can form a closed invariant curve. An example of such a curve is shown in Fig. 2(a) for the 1 : 5 tongue.

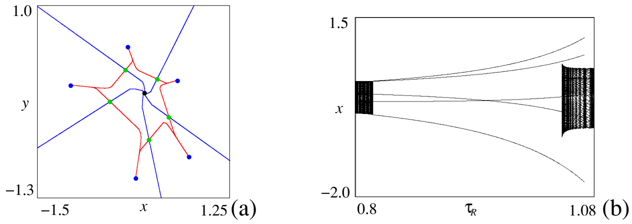

Figure 2: (a) Structure of the stable (blue) and unstable (red) manifolds. The stable period-5 cycle is denoted by the blue circles while the saddle period-5 cycle is denoted by the green circles. Here $\tau_L = 0.30$, $\tau_R = 1.0$, $\delta_L = 0.5$, $\delta_R = 1.6$, and $\mu = 0.05$. and (b) Bifurcation diagram with τ_R as the variable parameter, with the other parameters fixed at $\tau_L = 0.30$, $\delta_L = 0.5$, $\delta_R = 1.6$, and $\mu = 0.05$.

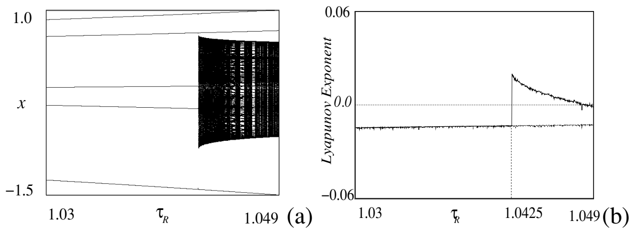

Figure 3: (a) Magnified part of the bifurcation diagram shown in (Fig. 2(b)). and (b) Largest Lyapunov exponent with $\tau_L = 0.30$, $\delta_L = 0.5$, $\delta_R = 1.6$, and $\mu = 0.05$.

Since our map is piecewise smooth, the stable and unstable manifolds are also piecewise smooth. As a result the closed invariant curve is piecewise smooth. If we increase or decrease the parameter τ_R, at critical values of the parameter, the stable and unstable fixed points collide and disappear through border collision fold bifurcation . To illustrate, Fig. 2(a) shows a period-5 attracting fixed point and a period-5 saddle fixed point forming a saddle-node connection at $\tau_R = 1.0$. As the parameter τ_R is increased, we have a parameter region where the period 5 attractor co-exists with a chaotic attractor (Fig. 2(b), Fig. 3(a)). The plot of the Lyapunov exponent in Fig. 3(b) shows that the orbit is indeed chaotic. From Fig. 4 we can see that the attractor is a circle-like strange attractor with infinite number of folds in it.

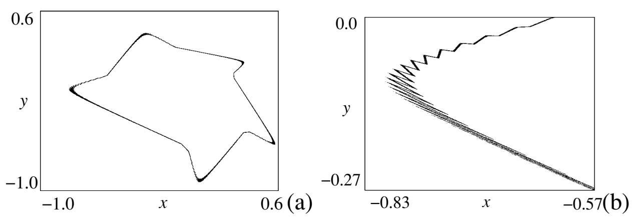

Figure 4: (a) A circle-like strange attractor for $\tau_L = 0.30$, $\tau_R = 1.0427$, $\delta_L = 0.5$, $\delta_R = 1.6$, and $\mu = 0.05$. (b) An enlarged portion of the strange attractor showing an infinite number of foldings.

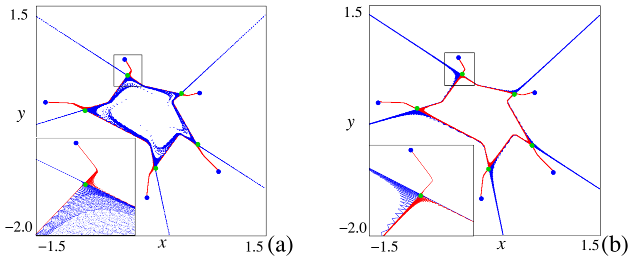

Figure 5: (a) The first homoclinic tangency between the stable (blue) and unstable (red) manifolds. The stable period-5 cycle is denoted by the blue circles while the saddle period-5 cycle is denoted by the green circles. Here $\tau_L = 0.30$, $\tau_R = 1.041812$, $\delta_L = 0.5$, $\delta_R = 1.6$, and $\mu = 0.05$. (b) The second homoclinic tangency between the stable (blue) and unstable (red) manifolds. Here $\tau_L = 0.30$, $\tau_R = 1.0425$, $\delta_L = 0.5$, $\delta_R = 1.6$, and $\mu = 0.05$.

4. Mechanism of the Occurrence of Chaos

To understand the sequence of bifurcations that lead to the chaotic orbit shown in Fig. 2(b) and Fig. 3(a), we start with $\tau_R = 1.0$ and gradually increase the parameter. At $\tau_R = 1.041812$, the manifolds become tangent to each other. This is called the first homoclinic tangency (Fig. 5(a)) and this leads to the formation of a non-transversal homoclinic orbit. With further change of τ_R, the stable and unstable manifolds intersect transversally to form a homoclinic structure. This implies the existence of a horseshoe. Next, at $\tau_R = 1.0425$, the unstable manifold moves to the right of the stable manifold and becomes tangent again. This is called the second homoclinic tangency (Fig. 5(b)). The strange attractor appears only after the second homoclinic tangency (Fig. 3(b)).

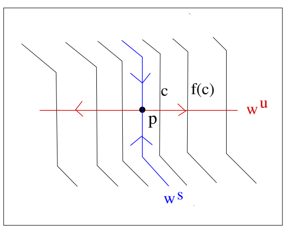

Figure 6: The stable and unstable manifolds along with the stable foliation of a saddle fixed point **p**.

The mechanism of generation of this chaotic orbit rests on the concept of stable foliation [2, 3]. Consider a saddle point **p** of a homeomorphism $\mathbf{f} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ (see the illustration in Fig. 6) . In the local neighborhood of the saddle point, we define a family of curves or "leaves" that satisfy the following properties:

- If **c** is any leaf, then **f(c)** is also contained in a leaf.
- Two points a and b lie on the same leaf iff $\mathbf{dist}(f(a)^n, f(b)^n)$ converges to zero exponentially fast.

The union of the family of curves satisfying the above properties is called a stable foliation. The stable manifold $\mathbf{W^s(p)}$ is a curve, which is also a leaf of the foliation.

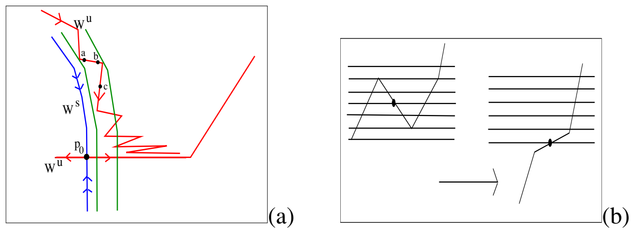

Figure 7: (a) Schematic diagram showing two quadratic tangencies of the unstable manifold with the leaves of the stable foliation (green) of a saddle point. (b) Schematic diagram showing the unfolding of two quadratic tangencies through an intersection with a leaf of the stable foliation.

Now consider one of the points of the period-5 saddle, which is a fixed point of the 5th iterate of the map. Consider a stable foliation in its neighborhood, shown schematically in Fig. 7(a). After the second homoclinic tangency, the unstable manifold generically has two nonsmooth quadratic tangencies with two leaves of the stable foliation. These tangencies are propagated by successive iterations of the map. Now consider three points a, b, and c on the unstable manifold as shown in Fig. 7. The distance between the successive images of the points b and c tends to zero exponentially while the distance between the points a and b increase at a slow asymptotic rate. As a result the unstable manifold, after returning very close to the saddle point, go forth and back indefinitely. This creates an infinite number of folds in the unstable manifold. As a result the closure of the unstable manifold would have infinite length. This gives rise to a large circular strange attractor (Fig. 4).

With further change in the parameter, two consecutive nonsmooth quadratic tangencies unfold through an intersection with the stable foliation as shown in Fig. 7(b). As more and more of such quadratic tangencies unfold, that stretch of the unstable manifold ceases to have the repeated foldings. Thus the length of the closure of the unstable manifold reduces progressively, and becomes finite at $\tau_R \approx 1.0483$. At this point the strange attractor disappears.

After the disappearance of the strange attractor on the closure of the unstable manifold, a high-periodic resonance tongue comes into existence through a nonsmooth saddle-node bifurcation and coexists with the period-5 tongue. Throughout the parameter region $1.0483 < \tau_R < 1.06878$, various high periodic resonance tongues coexist with the period-5 tongue. Finally at $\tau_R \approx 1.06878$ the period-5 attractor disappears through a border collision fold bifurcation.

We have found that the mechanism described above is generic. If a resonant torus goes through homoclinic bifurcation, a circular-shaped strange attractor with an infinite number of nonsmooth folds appears just after the second homoclinic contact.

5. Ergodic torus

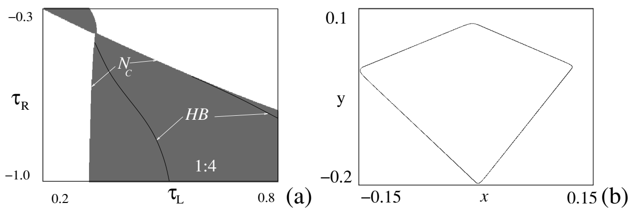

Figure 8: (a) The bottom part of the 1 : 4 resonance tongue. HB denotes the homoclinic bifurcation curves (the second homoclinic contact), N_C denotes the border-collision fold bifurcation curves. (b) The quasiperiodic attractor for $\tau_L = 0.33375$, $\tau_R = -0.39$, $\delta_L = 0.5$, $\delta_R = 1.6$, and $\mu = 0.05$.

If the orbit turns chaotic following the second homoclinic tangency, what is the mechanism of occurrence of an ergodic torus? Our investigation reveals that if a resonance torus does not break down through period doubling, it disappears through border collision fold bifurcation. In that case the resonance torus may not undergo homoclinic bifurcation. If a resonance torus does not go through homoclinic intersection, its border collision fold bifurcation will give rise to an ergodic torus because of the absence of any other high-periodic orbit. Fig. 8(a) shows a zoomed portion of the 1 : 4 resonance tongue, along with the homoclinic bifurcation curves. It is clear that there exists certain parameter regions where a resonance torus does not experience homoclinic bifurcation as a parameter is varied. For example if we fix τ_R at -0.39 and increase τ_L from 0.32, we observe the transition from a resonant torus to an ergodic torus after the border collision fold bifurcation at $\tau_L = 0.33375$ (see Figs. 8(b)).

6. Conclusion

The destruction of two-frequency torus is one of the classical routes to chaos. Earlier work had revealed a mechanism of torus destruction in piecewise smooth maps, which goes through the sequence: first homoclinic contact followed by homoclinic intersection, which is again followed by a second homoclinic contact. In this work we have shown that after the second homoclinic contact, a circular-shaped strange attractor with an infinite number of nonsmooth folds is created. The mechanism of this chaotic behavior is explained in terms of tangencies with the stable foliation of the saddle fixed point. In case the closed invariant curve does not undergo a homoclinic intersection, a quasiperiodic orbit exists when the invariant curve disappears through a border collision fold bifurcation.

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