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Optimal Phase Response Curves for Stochastic Synchronization and Desynchronization of Limit-cycle Oscillators

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Abstract—Optimization of phase response curves for stochastic synchronization and desynchronization induced by common Poisson noise is considered. By solving the Euler-Lagrange equation giving the extremum of the Lyapunov exponent, the optimal phase response curve for stochastic synchronization and desynchronization are obtained. We show numerical examples of the stochastic synchronization and desynchronization with the optimal PRCs.

1. Introduction

Synchronization phenomena of rhythmic elements attract much attention and are extensively studied in diverse fields [1]-[7]. Recently, it has been reported that synchronization of uncoupled oscillators is induced by common random driving signals such as Gaussian [9][8] and Poisson noise [10]-[12] . This phenomena, termed *stochastic* or *noise-induced synchronization*, may explain synchronous behavior of various systems ranging from lasers [3] and electric circuits [4] to spiking neurons [5][6] and ecological populations [7].

Stability of the synchrony is often quantified by the Lyapunov exponent, which measures the mean exponential growth rate of small difference between two oscillators states. For the case that the oscillators are driven by weak Gaussian noise, it was shown by Teramae and Tanaka that the Lyapunov exponent takes negative values and synchrony is induced for arbitrary phase response curves [9]. Abouzeid and Ermentrout obtained the optimal shape of the phase response curve (PRC) for stochastic synchronization by minimizing the Lyapunov exponent, which was nearly sinusoidal [8].

This prompts two questions. (i) What is the optimal shape of the PRCs to yield the most efficient stochastic synchronization for other driving signals? (ii) Are there any PRCs that induce stochastic desynchronization if the driving signal is not weak Gaussian?

Here we consider the case that the oscillators are driven by common Poisson noise. By solving the Euler-Lagrange equation, we calculate the PRCs with which the Lyapunov exponent takes extremum. We show that the optimal PRC for stochastic synchronization mutates from a sinusoid to a sawtooth by increasing its squared amplitude. We also show that stochastic desynchronization can occur for Poisson driving case and calculate the optimal PRC for that. Synchronization and desynchronization processes of the oscillators with the optimal PRCs are demonstrated by numerical simulations.

2. Stochastic synchronization and desynchronization

2.1. Poisson-driven oscillators

Let us consider that a pair of uncoupled oscillators is driven by common Poisson impulsive noise with constant amplitude:

$$\begin{aligned}\dot{\theta}_1(t) &= \omega + \sum_{n=1}^{N(t)} G(\theta_1)\delta(t-t_n),\\ \dot{\theta}_2(t) &= \omega + \sum_{n=1}^{N(t)} G(\theta_2)\delta(t-t_n),\end{aligned} \tag{1}$$

where $\theta_{1,2} \in [0, 1)$ are phase variables of the oscillators, ω is their natural frequency, $N(t)$ is a Poisson process of rate λ, $\{t_1, t_2, \cdots\}$ are arrival times of the Poisson impulses, and $G(\theta)$ is the PRC of the oscillators.

2.2. Lyapunov exponent

The Lyapunov exponent Λ, which quantifies the exponential growth rate of small phase differences between the oscillators $\Delta\theta(t) = \theta_1(t) - \theta_2(t)$, is given in terms of the PRC as

$$\Lambda = \lambda \int_0^1 d\theta P(\theta) \ln \left|1 + G'(\theta)\right|, \tag{2}$$

where $P(\theta)$ is a stationary PDF of the phase θ given by a stationary solution of the Frobenius-Perron equation corresponding to Eq. (1) [10]-[12].

We assume that the impulses are sparse, i.e., the Poisson rate λ is small. In this case, $P(\theta)$ can be approximated to be uniform, namely, $P(\theta) \simeq 1$ and the Lyapunov exponent is approximately given by [10][11]

$$\Lambda = \lambda \int_0^1 d\theta \ln \left|1 + G'(\theta)\right|. \tag{3}$$

2.3. Euler-Lagrange equation

We try to obtain the optimal shape of $G(\theta)$ for synchronization [or desynchronization] by minimizing [or maximizing] Λ with the constraint on its squared amplitude,

$$K[G] = \int_0^1 G(\theta)^2 d\theta - A = 0, \tag{4}$$

and examine the dependence of the optimal PRC on the parameter A that controls the squared amplitude. This constraint excludes the possibility of divergent $G(\theta)$ yielding arbitrarily negative or positive Lyapunov exponents, which is non-physical.

By Lagrange multiplier method, the optimal PRCs are found with which the following action $S[G]$ takes the extremum:

$$\begin{aligned} S[G] &= \Lambda[G] + \mu K[G] && (5) \\ &= \int_0^1 \left\{ \lambda \ln \left| 1 + G'(\theta) \right| + \mu \left(G(\theta)^2 - A \right) \right\} d\theta && (6) \\ &= \int_0^1 L(G(\theta), G'(\theta)) d\theta, && (7) \end{aligned}$$

where μ is a Lagrange multiplier, and $L(G(\theta), G'(\theta))$ is a Lagrangian. The corresponding Euler-Lagrange equation is given by

$$\frac{d}{d\theta}\frac{\partial L}{\partial G'} - \frac{\partial L}{\partial G} = 0. \tag{8}$$

$$\Rightarrow \quad G''(\theta) = -\frac{2\mu}{\lambda} G(1 + G')^2. \tag{9}$$

By solving this equation, we obtain the optimal PRCs for stochastic synchronization and desynchronization.

3. Optimal PRCs for Poisson driving case

3.1. Stochastic synchronization

If the parameter A, that gives the squared amplitude of the PRC, is sufficiently small, the PRC can be approximated as $G(\theta) = cZ(\theta)$, where $Z(\theta)$ is the phase sensitivity function [13]. Substituting this into equation (9) and taking the $c \to 0$ limit, we obtain the Euler-Lagrange equation

$$Z''(\theta) = -\frac{2c^2\mu}{\lambda} Z, \tag{10}$$

which yields sinusoidal $Z(\theta)$ for $\mu > 0$ and is consistent with the previous work [8]. On the other hand, if we ignore the constraint by the parameter A, $G(\theta) = -\theta + const.$ is a trivial solution to equation (9), which gives a sawtooth. Thus, when the squared amplitude of $G(\theta)$ is controlled, mutation of the optimal PRC between the two limiting shapes is expected. To confirm this, we numerically calculated a family of solutions to equation (9) using the shooting method [14].

When the Lagrange multiplier $\mu > 0$, the optimal PRCs for synchronization are obtained. Figure 1(a) shows the results, where the optimal solutions are wrapped within the range $[-0.5, 0.5)$ by taking modulo 1. All solutions lay within the plotted region, and no other solutions outside of this region were found. As expected, we see that the optimal PRC is almost sinusoidal when the parameter A is small. As A is increased, the optimal PRC gradually deviates from the sinusoid and approaches a symmetric sawtooth shape. Correspondingly, the Lyapunov exponent Λ plotted in the lower-right panel becomes more negative and tends to diverge.

Numerical examples of the stochastic synchronization process with the optimal PRCs are shown in figure 1(b)-(d). 30 oscillators are driven by common Poisson impulses of rate $\lambda = 0.2$. The natural frequency of the oscillators is fixed at $\omega = 1.0$. In each panel, the Lyapunov exponent calculated from equation (3) is shown at the top. As we see, the more the Lyapunov exponent gets negative, the more stochastic synchronization is facilitated.

3.2. Stochastic desynchronization

Are there PRCs that lead to desynchrionization? No such PRC exists for weak Gaussian driving case, or equivalently, for the case that the parameter A is sufficient small [9]. If the impulse is not that weak, however, there exist PRCs that yields positive Lyapunov exponent [11]. What shape is the optimal for desynchronization?

We varied the Lagrange multiplier in the negative range, $\mu < 0$, and found the optimal PRCs for desynchronization. As shown in figure 2(a), optimal PRC exhibits a sharp cusp at $\theta = 0$ when μ is sufficiently negative, and gradually approaches to a sawtooth as μ increases. The Lyapunov exponent gradually increases as we increase the parameter A [Uppre-right window in Figure 2(a)].

Numerical examples of the stochastic desynchronization with the optimal PRCs are shown in figure 2(b)-(d).. Numerical setup of the common Poisson noise and the natural frequency of each oscillator is the same as in figure 1. The Lyapunov exponent calculated from equation (3) is shown at the top of each panel. We see that as the Lyapunov exponent gets larger, the more the stochastic desynchronization is facilitated.

4. Summary

We considered the optimization problem of the PRC for stochastic synchronization and desynchronization of limit-cycles oscillators by common Poisson noise. Using the Lagrange multiplier method, we sought the PRCs that gave extremum of the Lyapunov exponent. We found that the optimal PRC for stochastic synchronization mutated from sinusoidal to sawtoothed by increasing its squared amplitude. We also found the optimal PRC for desynchronization. By numerical simulations with the optimal

PRCs, stochastic synchronization and desynchronization were demonstrated.

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References

[1] A. T. Winfree, *The Geometry of Biological Time* (Springer, New York, 2001)

[2] A. Pikovsky, M. Rosenblum, and J. Kurths, *Synchronization* (Cambridge University Press, England, 2001).

[3] A. Uchida, R. McAllister, and R. Roy, Phys. Rev. Lett. **93**, 244102 (2004).

[4] K. Yoshida, K. Sato, and A. Sugamata, J. Sound and Vibration **290**, 34 (2006).

[5] Z. F. Mainen and T. J. Sejnowski, Science **268**, 1503 (1995).

[6] R. F Galán, N. Fourcaud-Trocmé, G. B. Ermentrout, and N. N. Urban, J. Neurosci. **26**, 3646 (2006); R. F. Galán, G. B. Ermentrout and N. N. Urban, Sensors and Actuators B: Chemical. **116**(1-2), 168 (2006); R. F. Galán, G. B. Ermentrout and N. N. Urban, J. Neurophysiol. **99**, 277 (2008); G. B. Ermentrout, R. F. Galán, and N. N. Urban, Trends in Neurosciences **31**, 428 (2008).

[7] E. Ranta, V. Kaitala and E. Helle, Oikos **78**, 136 (1997).

[8] A. Abouzeid and G. B. Ermentrout, Phys. Rev. E **80**, 011911 (2009).

[9] J. Teramae and D. Tanaka, Phys. Rev. Lett. **93**, 204103 (2004).

[10] H. Nakao, K. Arai, K. Nagai, Y. Tsubo, and Y. Kuramoto, Phys. Rev. E **72**, 026220 (2005).

[11] K. Arai and H. Nakao, Phys. Rev. E **78**, 066220 (2008).

[12] S. Hata, T. Shimokawa, K. Arai, and H. Nakao, Phys. Rev. E **82**, 036206 (2010).

[13] Y. Kuramoto, *Chemical Oscillation, Waves, and Turbulence* (Springer-Verlag, Tokyo, 1984) (republished by Dover, New York, 2003).

[14] W. H. Press, B. P. Flannery, S. A. Teukolsky, and W. T. Vetterling, *Numerical Recipes 3rd Edition: The Art of Scientific Computing* (Cambridge University Press, 2007).

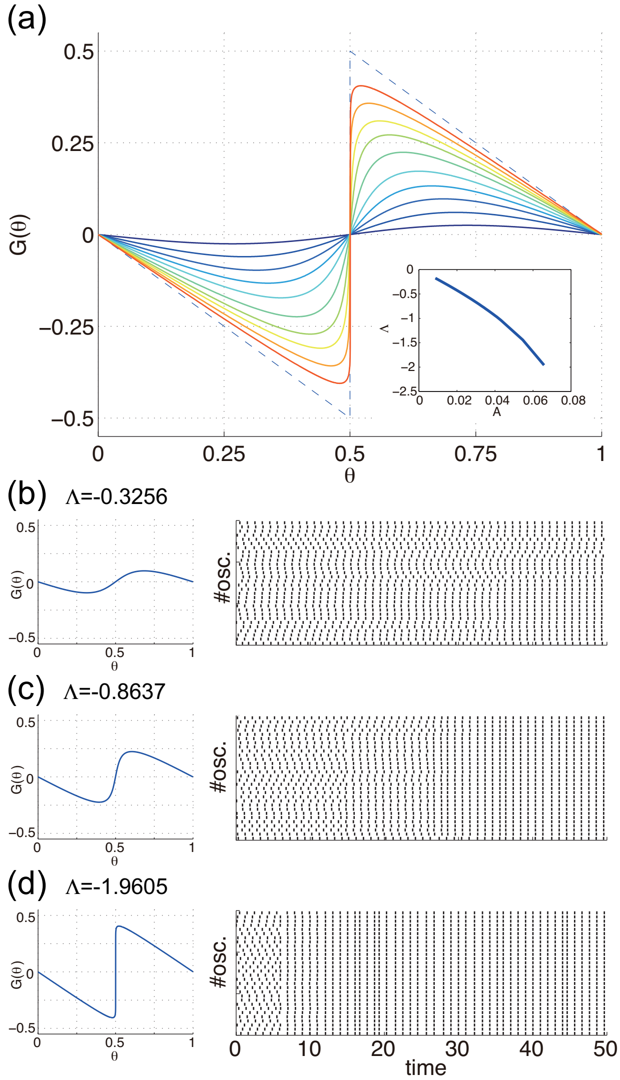

Figure 1: Optimal PRCs for stochastic synchronization. (a) Numerical solutions of the Euler-Lagrange equation (9) obtained by the shooting method. The dashed line shows the limiting sawtooth solution. The inset plots the dependence of the Lyapunov exponent Λ on the parameter A. (b)-(d) Numerical realizations of the stochastic synchronization process with the optimal PRCs. In each panel, the corresponding PRC is shown in the left panel.

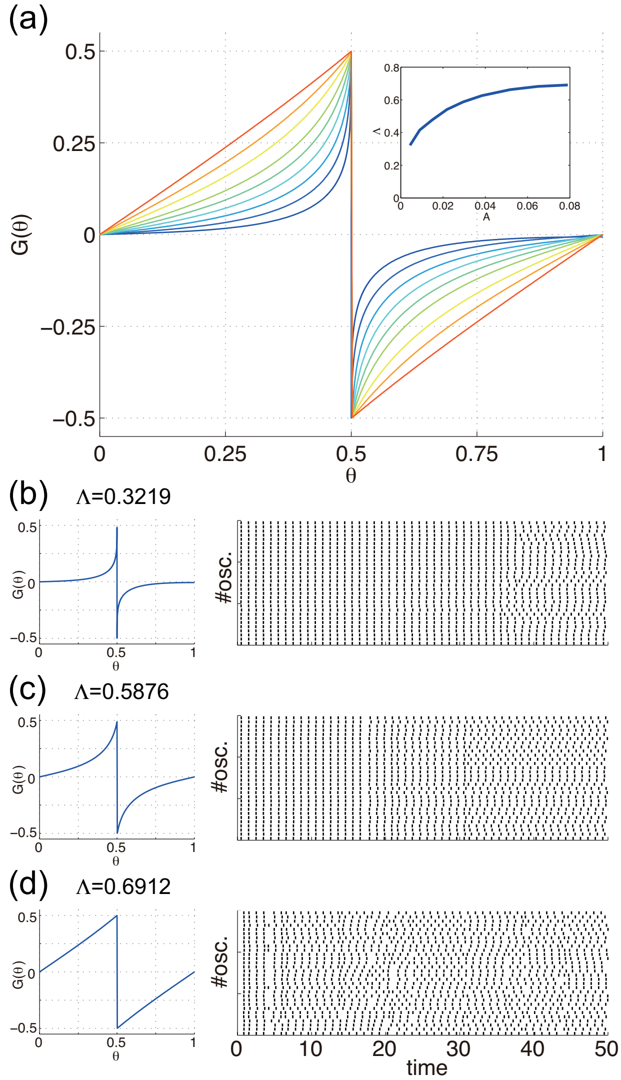

Figure 2: Optimal PRCs for stochastic desynchronization. (a) Numerical solutions of the Euler-Lagrange equation (9) obtained by the shooting method. Small inset at the lower-right plots the dependence of the Lyapunov exponent Λ on the parameter A. (b)-(d) Numerical demonstrations of the stochastic desynchronization process with the optimal PRCs. In each panel, the corresponding PRC is shown in the left panel.