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Variable Resolution Gabor Transform Using The Discrete Convolution of Gaussian in The Frequency Domain

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Abstract—Gabor Transform(GT) is often used to time-frequency analysis for non-stationary signal. The time-frequency analysis including GT can't simultaneously improve time resolution and frequency resolution. For this problem, various studies of multiple resolution analysis have been done. In these studies, the calculation complexity got very high because several resolution analysis must be performed in the same signal. Solving this issue, we have proposed the resolution conversion method which can synthesize the spectrum of high frequency resolution using some spectra of low frequency resolution. However, the method can just convert the spectrum to higher frequency resolution. In this paper, we propose a conversion method to lower frequency resolution. The proposed method allows you to get arbitrary resolution spectrum in any time-frequency position by synthesizing the desired one from the spectra calculated preliminarily in one resolution.

1. Introduction

Short Time Fourier Transform(STFT) is often used to time-frequency analysis for non-stationary signal. In this method, it performs a Fourier Transform(FT) to what was cut out a part of signal as the frame. In general, in order to ensure continuity at the endpoints of the frame, eliminate the influence of the discontinuity by windowing. Result of the frequency analysis is different depending on whether choose what kind of window function. Gabor Transform(GT) to be used Gabor function which is famous for that the product of the time-frequency resolution is minimum[1] is proposed. GT have been used in a variety of fields[2], but the analysis resolution is bound by the uncertainty principle, time resolution and frequency resolution has a relationship of trade-off [3]. In this effect, there is a problem which the time-frequency analysis can't improve time resolution and frequency resolution at the same time.

For this problem, a study of multiple resolution analysis has been done[4]. In this method, it perform the GT of some different frame length for the input signal, and to select the best resolution at a certain time and frequency. However, in this study, the calculation complexity got very high because several resolution analysis must be performed in the same signal. Solving this issue, we have proposed the resolution conversion method which can synthesize the spectrum of high frequency resolution using some spectra of low frequency resolution[5]. However, it can synthesize in only one direction to a spectrum of high frequency resolution from spectra of low frequency resolution in this method. In this paper, we propose the method for synthesizing a high time resolution spectrum from low time resolution spectrum. The proposed method allows you to get arbitrary resolution spectrum in any time-frequency position by synthesizing the desired one from the spectra calculated preliminarily in one resolution. First, we show it can be synthesized a high time resolution spectrum from a low time resolution spectrum using mathematical characteristics of Gaussian that convolution of two Gaussians become a Gaussian with different standard deviation. Next, we discuss of the calculation accuracy of the proposed method, and show a design method of synthesis based on the calculation accuracy. Finally, we experiment and show a high time resolution spectrum can be synthesized from low time resolution spectra, and compare the calclation complexity, we showed the practical value of the proposed method.

2. Gabor Transform

GT $X_\sigma(\tau,\omega)$ of input signal $x(t)$ is defined by

$$X_\sigma(\tau,\omega) = \int g_\sigma(t-\tau)\cdot x(t)\cdot e^{-j\omega(t-\tau)}dt \tag{1}$$

where Gauss window $g_\sigma(t)$ is defined by

$$g_\sigma(t) = \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{t^2}{2\sigma^2}}. \tag{2}$$

Shape of the window, that is to say the standard deviation σ of the Gauss window determines a resolution. Since Gaussian is a function of infinite length, an error occurs by truncating a finite length for use Gaussian as a window function. Therefore, we design a sigma that the amplitude of Gaussian which is a monotonically decreasing function is below the acceptable error ε at the end point of the frame.

$$\sigma = \frac{1}{2\sqrt{-2\ln\varepsilon}}\cdot\frac{L}{f_s} \tag{3}$$

where L is frame length, f_s is sampling frequency.

3. Synthesis Method

In general, in order to obtain a spectrum of different resolutions, it is necessary to perform GT again. We propose a method that directly calculate a spectrum of σ/α from a spectrum which is calculated by the Gauss window of σ in this paper. Where α is synthetic magnification. Thus, we use the property which frequency response of a Gaussian is a Gaussian. FT of Eq.(2) is

$$G_\sigma(\omega) = \int g_\sigma(t)\cdot e^{-j\omega t}dt = e^{-\frac{\omega^2}{2(\frac{1}{\sigma})^2}}. \quad (4)$$

In order to operate σ of GT using this property, we focused on the mathematical characteristics of Gaussian that convolution of two Gaussians become a Gaussian with different standard deviation.

$$G_{\frac{1}{\alpha}\sigma}(\omega) = \int G_\sigma(\omega-\gamma)\cdot G_{\sigma_d}(\gamma)d\gamma \quad (5)$$

where σ_d is

$$\sigma_d = \frac{\sigma}{\sqrt{\alpha^2-1}} \quad (\alpha > 1). \quad (6)$$

Substituting Eq.(5) to the GT , Eq.(7) is obtained undergoing a process of follow.

$$\begin{aligned} X_{\frac{1}{\alpha}\sigma}(\tau,\omega_0) &= e^{j\omega_0\tau}\int g_{\frac{1}{\alpha}\sigma}(t-\tau)\cdot x(t)\cdot e^{-j\omega_0 t}dt \\ &= e^{j\omega_0\tau}\int G_{\frac{1}{\alpha}\sigma}(\omega)\cdot X(\omega+\omega_0)\cdot e^{j\omega\tau}d\omega \\ &= e^{j\omega_0\tau}\int\int G_\sigma(\omega-\gamma)\cdot X(\omega+\omega_0)\cdot e^{j(\omega-\gamma)}d\omega \\ &\quad \cdot G_{\sigma_d}(\gamma)\cdot e^{j\gamma\tau}d\gamma \\ &= \int e^{j\gamma\tau}\cdot X_\sigma(\tau,\omega_0+\gamma)\cdot G_{\sigma_d}(\gamma)d\gamma \end{aligned} \quad (7)$$

This expression means that as shown in Figure 1, a spectrum of different σ can be calculated by convoluting Gaussian $G_{\sigma_d}(\gamma)$ with multiplying a spectrum $X_\sigma(\tau,\omega_0+\gamma)$ by a phase term $e^{j\gamma\tau}$ in the frequency domain.

Here, the frequency domain is discretized at intervals $\Omega = \frac{\omega_s}{L}$ by DFT. Where ω_s is sampling angular frequency. In order to perform product-sum operation to the convolution of Eq.(7) , as $\gamma = m\Omega$, it is assumed that convolving to the point of finite term M_ω, the discrete formula is

$$\widetilde{X}_{\frac{1}{\alpha}\sigma}(\tau,\omega_0) \simeq \sum_{m=-M_\omega}^{M_\omega}\{X_\sigma(\tau,\omega_0+m\Omega)\cdot G_{\sigma_d}(m\Omega)\cdot e^{-j(m\Omega)\tau}\}\cdot\Omega. \quad (8)$$

4. Evaluate of Synthetic Accuracy

A spectrum can be synthesized without the error in the continuum region, but the error occurs by discretizing as

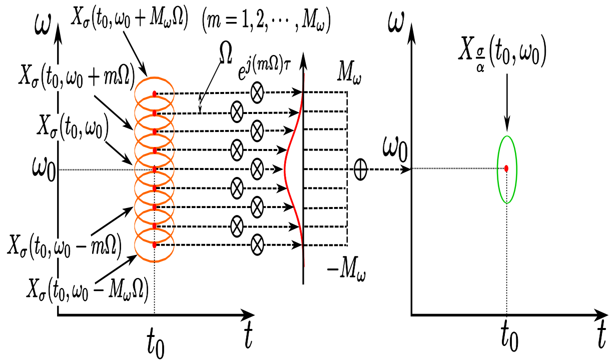

Figure 1: Spectrum synthesis method

Eq.(8). The synthesizing of a spectrum is performed by convolving spectrum which is calculated with respect to each interval Ω in frequency domain to the point of finite term M_ω. Thus, we discuss how M_ω and Ω affect to the error.

First, we discuss the error of synthetic Guass window in the time domain. Eq.(8) is rewritten in the form of the GT using the frequency response of the synthetic Gauss window as Eq.(9).

$$\begin{aligned} X_{\frac{1}{\alpha}\sigma}(\tau,\omega_0) &= \sum_{m=-M_\omega}^{M_\omega}\{e^{j(\omega_0+m\Omega)\tau}\int X(\omega)\cdot G_\sigma(\omega-\omega_0-m\Omega) \\ &\quad \cdot e^{j\omega\tau}d\omega\cdot G_{\sigma_d}(m\Omega)\cdot e^{-j(m\Omega)\tau}\}\cdot\Omega \\ &= e^{j\omega_0\tau}\int \widetilde{G}_{\frac{1}{\alpha}\sigma}(\omega)\cdot X(\omega+\omega_0)\cdot e^{j\omega\tau}d\omega \end{aligned} \quad (9)$$

In other word, the error of synthetic spectrum is controlled by the error of frequency response of synthetic Gauss window $\widetilde{G}_{\frac{1}{\alpha}\sigma}(\omega)$. Since convolution in the frequency domain is the product of the time domain, $\widetilde{G}_{\frac{1}{\alpha}\sigma}(\omega)$ becomes the product as Eq.(10) in the time domain.

$$\widetilde{g}_{\frac{1}{\alpha}\sigma}(t) = g_\sigma(t)\cdot g_{\sigma_d}(t-\frac{2\pi k}{\Omega}) \quad (10)$$

Gaussian which is a window function of the GT is a monotonically decreasing function, but as shown in Figure2, a discrete Gaussian becomes periodic. In this effect, we can be seen that the synthetic Gauss window $\widetilde{g}_{\frac{1}{\alpha}\sigma_d}(t)$ has a periodic component. Where, k is a parameter which represent what number periods of periodic components of the discrete Gaussian. The peak level of the first periodic component controls the accuracy of the synthesis Gauss window. Hence, we calculate the level of the first periodic component which is occurred in $k = 1$. Using Eq.(10), synthetic Gauss window can be written as

$$\widetilde{g}_{\frac{1}{\alpha}}(t) = e^{-\frac{1}{2}\left\{\left(\frac{t}{\sigma}\right)^2+\left(\frac{t-\frac{2\pi}{\Omega}}{\sigma_d}\right)^2\right\}}. \quad (11)$$

Furthermore, since the derivative of periodic component at peak time t_p is 0, the result of differentiating $\widetilde{g}_{\frac{1}{\alpha}\sigma}(t_p)$ with

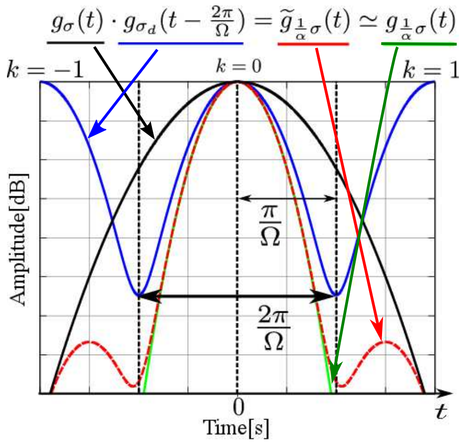

Figure 2: Characteristics of the synthetic Gauss window in the time domain

respect to t_p is 0, and solving for t_p,

$$t_p = \frac{2\pi}{\Omega}\left(1 - \frac{1}{\alpha^2}\right) \tag{12}$$

By substituting this t_p to Eq.(11) of synthetic Gauss window, the peak of the periodic component is obtained.

$$\widetilde{g}_{\alpha\sigma}\left(t_p\right) = e^{-\frac{1}{2}\left(\frac{1}{\sigma}\right)^2\left(\frac{2\pi}{\Omega}\right)^2\left(1-\frac{1}{\alpha^2}\right)} \tag{13}$$

It can be seen that the peak of the periodic component does not depend on the frequency of analysis from this formula. Also, σ and α are determined by the request of the user. Thus, it is necessary to design the ω as the peak of periodic component drops below the acceptable error.

Next, we discuss the relationship between the interval Ω of frequency domain and the error. Thus, we define the required accuracy of the synthesis depending on the analysis object, and determine the longest intervals Ω satisfying the required precision. If the level of periodic component which is required is ε, from $\widetilde{g}_{\frac{1}{\alpha}\sigma}(t_p) \leq \varepsilon$, Ω is

$$\Omega \leq \left|\frac{\sqrt{2}\pi}{\sqrt{-\ln \varepsilon}\sigma} \cdot \sqrt{1 - \frac{1}{\alpha^2}}\right|. \tag{14}$$

However, Ω take only regular intervals by DFT. If substituting $\Omega = \frac{\omega_s}{L}$ and Eq.(3), Eq.(14) is

$$\alpha \geq \sqrt{\frac{4}{3}}. \tag{15}$$

Finally, we discuss the relationship between the number of synthesis term M_ω and the error. To determine the minimum M_ω satisfying the condition by defining the accuracy of the request in the same manner as Ω. Thus, it is considered to be sufficient to consider the range which amplitude of discrete Gaussian is damped to required accuracy. Solving for ω under the conditions that frequency response of the discrete Gaussian $G_{\sigma_d}(\omega)$ is less than the ε, using $\omega = M_\omega\Omega$,

$$M_\omega\Omega \geq \frac{\sqrt{-2\ln\varepsilon}\sqrt{\alpha^2 - 1}}{\sigma}. \tag{16}$$

If substituting $\Omega = \frac{\omega_s}{L}$ and Eq.(3) to this fomula,

$$M_\omega \geq \frac{-2\ln\varepsilon\sqrt{\alpha^2 - 1}}{\pi}. \tag{17}$$

As above, the condition satisfying required accuracy of the intervals Ω and the number of synthesis term M_ω becomes clear. By satisfying this condition, a spectrum of the high time resolution can be synthesized within a tolerance from spectra of the low time resolution .

5. Experiment

It was confirmed whether capable of synthesizing a spectrum of the high time resolution from spectra of the low time resolution in practice. We conducted experiments to three sine waves. We are shown the parameters of experiment in Table1. Figure 3 is an original spectrogram, and figure 4 is a spectrogram of after synthesis. Figures are represent, the horizontal axis is time, the vertical axis is frequency, and the brightness is amplitude. We could confirm that can be synthesized a spectrum of high time resolution from spectra of low time resolution from experimental results.

Table 1: The parameters of experiment

Input Signal : sine wave	440,660,880[Hz]
Frame Length : L	16384[sample]
Sampling Frequency : f_s	44.1[kHz]
Acceptable Error : ε	$3.05 \cdot 10^{-5}$
Synthetic Magnification : α	2 (Namely $\frac{1}{2}\sigma$)

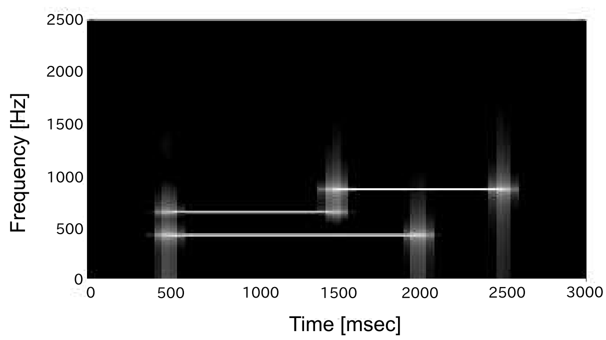

Figure 3: Original spectrogram

The proposed method can calculate an approximation of the Continuous Wavelet Transform (CWT) by changing the resolution for each frequency. Hence, we compare the calculation complexity of the proposed method

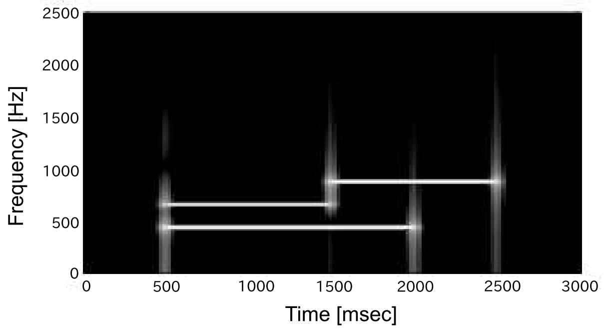

Figure 4: Synthetic spectrogram

O_p with the CWT O_w, and we consider whether there is a practical value. Comparing in number of multiplications that govern the calculation complexity. Specifically, We consider that scale up the scaling factor s from 1 to S. s can be written using integer a and interval $\Delta s = 0.5$. $s = a \cdot \Delta s + 1 (a : 0 \sim (S-1)/\Delta s)$. Since there is a condition $\alpha \geq \sqrt{4/3}$ in the proposed method, the resolution which is obtained by FFT at the beginning is matched with the resolution of frequency f_b of CWT. $f_b = f_0/(S \cdot \sqrt{4/3})$. We define $S' = S \cdot \sqrt{4/3}$. S' is integer. A formula of calculation complexity O_p of the proposed method is

$$O_p = 4(L \log_2 L) + L + \sum_{a=0}^{(S-1)/\Delta s} 4(2M_\omega + 1) \quad (18)$$
$$= 4(L \log_2 L) + L + \sum_{a=0}^{(S-1)/\Delta s} 4(2 \frac{-2 \ln \varepsilon \sqrt{\alpha^2 - 1}}{\pi} + 1)$$

where $\alpha = S'/(S - (a\Delta s))$. First term of Eq.(18) is calculation complexity of FFT. First term is quadrupled since calculating complex number. Second term is windowing. Third term is synthesis. Synthesis of single can calculate in the complex arithmetic of $(2M_\omega + 1)$. We compared using the calculation complexity of CWT O_w which is already calculated [6].

$$O_w = \sum_{a=0}^{(S-1)/\Delta s} 2(2\sqrt{-2\ln \varepsilon} f_s \sigma_w (a\Delta s + 1) + 1) \quad (19)$$

where $\sigma_w = L/(2\sqrt{-2\ln \varepsilon} \cdot f_s \cdot S')$. We show reduction rate $r = O_p/O_w$ in the figure 5. Where, $L = 16384$[sample], $f_s = 44.1$[kHz], $\varepsilon = 3.05 \cdot 10^{-5}$, $S = 32, 64, 128, 256$

We can reduce the calculation complexity to less than one-tenth compared with CWT when S = 512. Complexity reduction method of multiple resolution CWT[6] is expected to further reduce the calculation complexity using the proposed method.

6. Conclusion

In this paper, we designed to indicate that it can be synthesized a spectrum of the high time resolution from

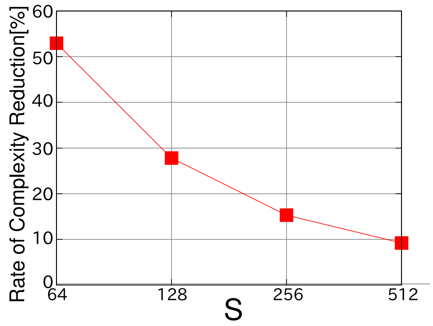

Figure 5: Rate of Complexity reduction

spectra of low time resolution. We focused on the mathematical characteristics of Gaussian that convolution of two Gaussians become a Gaussian with different standard deviation,and proposed the method for obtaining a spectrum of high time resolution by convolving the Gaussian in the frequency domain to the spectrum obtained by the GT. Also, we discussed the error which occurs in synthesizing the spectrum, and showed that by determining the accuracy required by the user, synthesis within a tolerance is possible. Furthermore, we experimented and showed it is possible to synthesize a spectrum of the high time resolution from spectra of low time resolution, and compare the calclation complexity, we showed the practical value of the proposed method.

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