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Limit Cycle Control for 2-Dimeisional Discrete-time Nonlinear Control Systems and Its Application to Chaotic Systems

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Abstract—In this paper, we propose a controller generating a desired limit cycle for a 2-dimensional discrete-time nonlinear control system. First, we state some notations and our problem setting. Next, we derive a control algorithm to solve the problem on limit cycles, and a modification of the algorithm is also shown. Then, we apply our algorithms to a chaotic system, the Hénon Map, to show the availability of the proposed method.

1. Introduction

The concept of limit cycles is quite important in various research fields such as stable walking or gait of humanoid robots in robot engineering, oscillator circuits in electronic engineering, catalytic hypercycles in chemistry, circadian rhythm in biology, boom-bust cycles in economics and so on. Phenomenon of limit cycles is specific for nonlinear systems, along with chaos and fractal, and it has been attracted a lot of researcher's interest. Therefore, researches on limit cycles have been vigorously done from mathematical and engineering perspectives so far.

In control theory, a lot of researchers have focused on synthesis problems of systems that generate limit cycles [1]. For example, in recent work, synthesis methods of nonlinear/hybrid systems whose solution trajectories converge to desired limit cycles are proposed in [2, 3, 4], and robust generation of oscillations for a class of nonlinear systems is studied in [5, 6]. On the other hand, there are few studies on design of only control inputs that realize a desired limit cycle for a given nonlinear control system. Since we design only control inputs to make the solution trajectories converge to the desired limit cycle, it seem to be quite difficult to solve the above synthesis problem. In [7], we have presented a control strategy based on the control Lyapunov function approach, which generates a desired limit cycle for 2-dimensional continuous-time nonlinear control systems. However, such a research for discrete-time nonlinear control systems has not been done.

In this paper, we propose a controller design method that generates desired limit cycles for 2-dimensional discrete-time nonlinear control systems. We first give some terminologies and state our problem. Next, two types of control algorithms that generates a desired limit cycles for the system are introduced. Then, we consider an application to a discrete-time chaotic system, the Hénon Map, to show the availability of the proposed control strategy.

2. Problem Setting

In this paper, we consider the following 2-dimensional discrete-time nonlinear control system defined in an open subset $D \subset \mathbf{R}^2$:

$$x_{k+1} = f(x_k) + g(x_k)u_k, \tag{1}$$

where $k = 0, 1, 2, \cdots \in \mathbf{Z}$ is a time step, $x_k = [\, x_k^1 \;\; x_k^2 \,]^\mathsf{T} \in D$ is a state variable, $u_k \in \mathbf{R}$ is a control input and $f,\ g : D \to TQ$ are smooth vector-valued functions defined in D.

We define *a limit cycle function* $V : D \to \mathbf{R}$:

$$V(x_k) := (x_k - s)^\mathsf{T} P(x_k - s) - r^2, \tag{2}$$

where $P \in \mathbf{R}^{2\times2}$ is a symmetric and positive definite constant matrix, $s \in \mathbf{R}^2$ is a constant vector and $r > 0$ is a constant. It can be easily confirmed that the equation $V(x_k) = 0$ determines a unique ellipse D_0 in D. We design P, s and r in (2) to make the ellipse the desired limit cycle. Note that D_0 can be set by rotating and translating a ellipse whose center is the origin, that is, P can be represented by a product of a rotating matrix R and a diagonal matrix T:

$$P = \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}_{R^\mathsf{T}} \underbrace{\begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix}}_{T} \underbrace{\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}}_{R}, \tag{3}$$

where θ is the rotating angle and $T_1,\ T_2 > 0$

Based on the problem setting above, we consider the following problem on limit cycle.

Problem 1: For the 2-dimensional discrete-time nonlinear control system (1), find a control strategy such that D_0 is an unique stable limit cycle of a solution trajectory of (1) from an initial state in D. ∎

3. Limit Cycle Control

3.1. Algorithm

In this subsection, we derive a control algorithm, which may be a solution of Problem 1. In order to achieve the desired limit cycle, a solution trajectory of (1) needs to converge to D_0. In other words, we have to find a control input such that $V(x_k)$ converges to 0 as time goes by, that is, $V(x_k)^2$ decreases as k increases. The following theorem gives us a necessary and sufficient condition on the existence of such a control input.

Theorem 1: For the 2-dimeinsional discrete-time nonlinear control system (1), a control input u_k that decreases the value of $V(x_k)^2$ exists if and only if

$$\begin{aligned}\Gamma(x_k) &:= \frac{b(x_k)^2}{a(x_k)} - c(x_k) + |V(x_k)| > 0,\\ a(x_k) &:= g(x_k)^{\mathsf{T}} P g(x_k),\\ b(x_k) &:= (f(x_k) - s)^{\mathsf{T}} P g(x_k),\\ c(x_k) &:= (f(x_k) - s)^{\mathsf{T}} P (f(x_k) - s) - r^2.\end{aligned} \tag{4}$$

holds. ∎

Since (4) is independent of k, so the domain such that (4) is satisfied can be calculated in advance. We call such a domain *a limit cycle basin of attractor of (1) for V*, and D_V denotes it. We have to set an initial state x_0 and a desired limit cycle D_0 so that they are included in D_V.

In order to generate a desired limit cycle for (1), a solution trajectory from x_0 always needs to be included in D_V. That is, we have to calculate a control input u_k such that $x_{k+1} \in D_V$ holds. Substituting (1) into $\Gamma(x_{k+1}) > 0$, we obtain an inequality with respect to u_k. By finding a u_k satisfying the inequality and apply it to (1), we can make $x_{k+1} \in D_V$. The procedure above is summarized as the following algorithm.

Algorithm 1:

(Step 0) Decide an initial states x_0 and a desired limit cycle D_0. Set $k = 1$.

(Step 1) Calculate u_k such that $\Gamma(x_{k+1}) > 0$ and apply it to (1). Replace k with $k + 1$ and continue this step. If u_k does not exist or a terminal condition is satisfied, this algorithm is finished. ∎

3.2. Modified Algorithm

By using Algorithm 1, it can be guaranteed that a solution trajectory of (1) starting from an initial state x_0 always satisfies $x_k \in D_V$ and converges to D_0. However, it is expected that as it converges to D_0, the range of available control input gets smaller and the solution trajectory does not behave a limit cycle. To overcome this problem, we define a new domain:

$$D_\varepsilon := \{\, x_k \in D \mid |V(x_k)| < \varepsilon \,\}. \tag{5}$$

If $x_k \notin D_\epsilon$, we adopt Step 1 of Algorithm 1. If $x_k \in D_\epsilon$, we calculate a control input u_k such that only $x_{k+1} \in D_\epsilon$ holds, that is, we do not consider convergence of the solution trajectory to D_0 and not use the condition $\Gamma(x_{k+1}) > 0$. We sum up a modification of Algorithm 1 as follows.

Algorithm 2:

(Step 0) Decide an initial states x_0 and a desired limit cycle D_0. Set $k = 1$.

(Step 1) Calculate u_k such that $\Gamma(x_{k+1}) > 0$, and apply it to (1). Replace k with $k + 1$ and continue this step. If $x_{k+1} \in D_\epsilon$ holds, then go to Step 2 with replacing $k+1$ with l. If u_k does not exist or a terminal condition is satisfied, this algorithm is finished. ∎

(Step 2) Calculate u_l such that $x_{l+1} \in D_\epsilon$, and apply it to (1). Replace l with $l + 1$ and continue this step. If u_l does not exist or a terminal condition is satisfied, this algorithm is finished. ∎

4. Application to Hénon Map

4.1. Probelm Setting

In this section, we consider an application of the limit cycle control method shown in the previous section to a discrete-time chaotic system called *the Hénon Map*. The autonomous Hénon Map is given by

$$\begin{bmatrix} x^1_{k+1} \\ x^2_{k+1} \end{bmatrix} = \begin{bmatrix} 1 - 1.4(x^1_k)^2 + x^2_k \\ 0.3x^1_k \end{bmatrix}, \tag{6}$$

where $x_k = [\, x^1_k \;\; x^2_k \,]^{\mathsf{T}} \in \mathbf{R}^2$ is a state variable. The solution trajectory from the initial state $x_0 = [\, 0.5 \;\; 0.1 \,]^{\mathsf{T}}$ is depicted in Fig. 1. This system shows some typical properties of chaotic systems such as strange attractor, fractal structure, sensitive dependence on initial conditions and so on. We add a control input $u_k \in \mathbf{R}$ to the second system of (6) as follows:

$$\begin{bmatrix} x^1_{k+1} \\ x^2_{k+1} \end{bmatrix} = \begin{bmatrix} 1 - 1.4(x^1_k)^2 + x^2_k \\ 0.3x^1_k \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_k. \tag{7}$$

Note that the control input cannot vanish the nonlinear term in the first system of (6), and so we do not consider an easy situation.

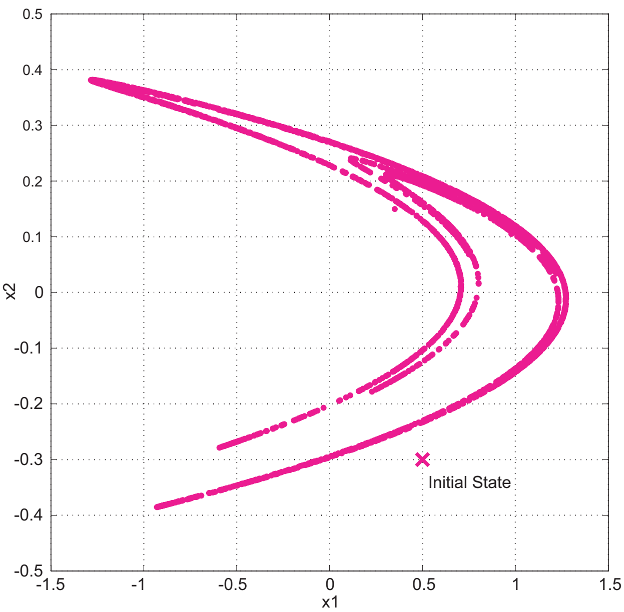

Fig. 1 : Hénon Map

Now, we set parameters of the limit cycle function. For parameters:

$$\begin{aligned}&\theta = 45^\circ,\; T_1 = 1,\; T_2 = \sqrt{2},\\ &s = [\, 0 \;\; 0 \,]^{\mathsf{T}}, r = 0.5,\end{aligned} \tag{8}$$

D_0 (the blue ellipse) and D_V (the green area) are illustrated in Fig. 2. From this figure, it can be confirmed that D_0 is not included in D_V, and hence these parameters cannot be used. Next, for other parameters:

$$\begin{aligned} &\theta = -60^\circ, T_1 = 1,\ T_2 = \sqrt{2}, \\ &s = [\,0 \quad -1\,]^{\mathsf{T}},\ r = 0.3, \end{aligned} \tag{9}$$

D_0 and D_V are illustrated in Fig. 3. From this figure, we can see that D_0 is included in D_V, and hence we adopt these parameters (9).

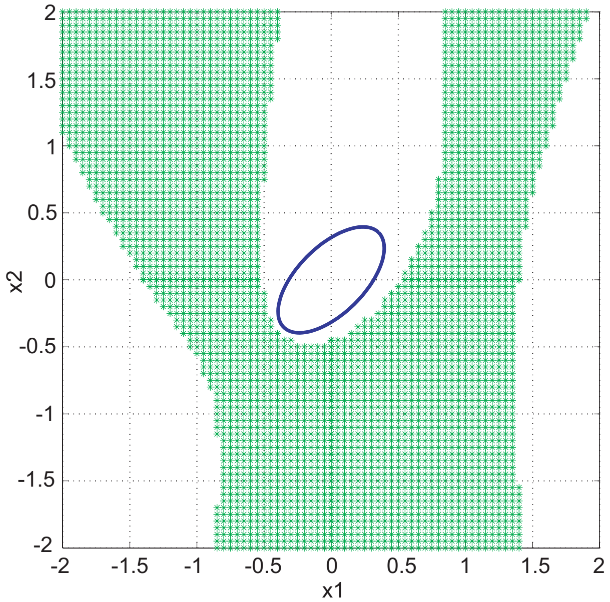

Fig. 2 : Limit Cycle and Basin of Attraction for (8)

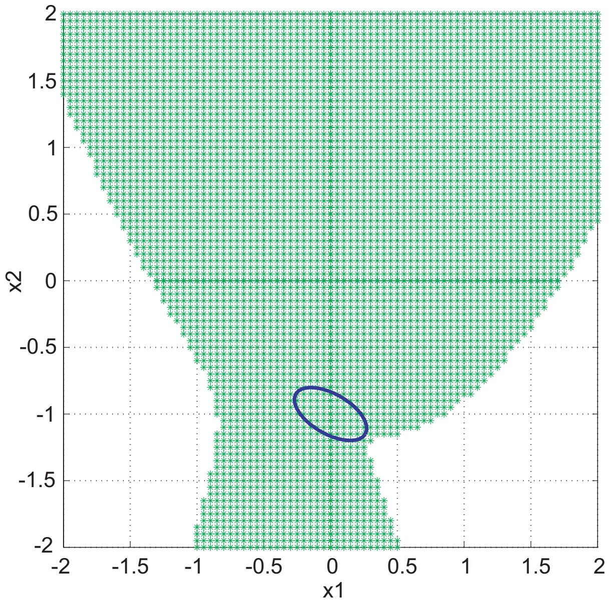

Fig. 3 : Limit Cycle and Basin of Attraction for (9)

4.2. Numerical Simulation

This subsection gives numerical simulations to check our algorithms in Section 3. First, we apply Algorithm 1 to (7) and (9). We set the initial state as $x_0 = [\,0.5 \quad -0.3\,]^{\mathsf{T}}$. Using Algorithm 1, we have the simulation result shown in Fig. 4. Fig. 4 illustrates the solution trajectory on the xy-plane. From this figure, it can be confirmed that the solution trajectory converges to D_0 by Algorithm 1. However, it distributes to only some parts of D_0.

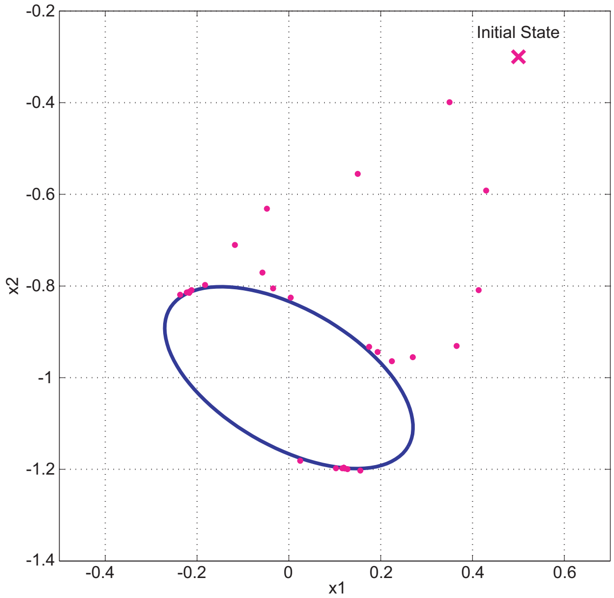

Fig. 4 : Solution Trajectory in xy-Plane (Algorithm 1)

Next, we use Algorithm 2 for (7) and (9). In Algorithm 2 We set $\epsilon = 0.02$ in (5). Fig. 5 shows the solution trajectory on the xy-plane. From this figure, we can confirm that the solution trajectory first converges to D_0 by Step 1 of Algorithm 2, then after the solution trajectory is included in D_ϵ, it distributes to most parts of D_0. Compared to Algorithm 1, Algorithm 2 shows a better performance of generating a limit cycle for the Hénon Map.

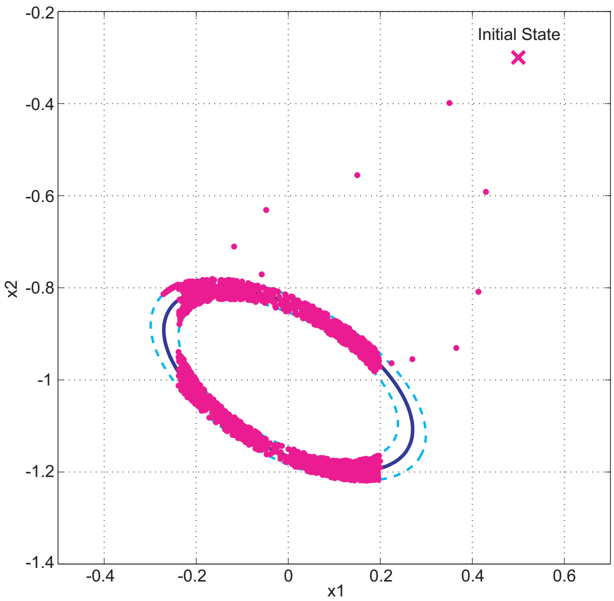

Fig. 5 : Solution Trajectory in xy-Plane (Algorithm 2)

5. Conclusion

In this paper, we have proposed an algorithm that generates a desired limit cycle for a 2-dimensional discrete-time nonlinear control system, and shown its modification. Then, we have applied the algorithms to a chaotic example, the Hénon Map, and indicated the effectiveness of our methods by numerical simulations.

Our future work is as follows: an extension of the proposed algorithms to multi-dimensional systems and applications to other systems.

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