2009 International Symposium on Nonlinear Theory and its Applications
NOLTA'09, Sapporo, Japan, October 18-21, 2009

Stabilization Control and Experiments of the Cart-Pendulum based on Discrete Mechanics

Tatsuya KAI† and Takeshi SHINTANI†

†Dept. of Mechanical Engineering, Grad. School of Engineering, Osaka University, JAPAN
Email: {kai, sintani}@watt.mech.eng.osaka-u.ac.jp

Abstract—This paper considers the stabilization problem of the cart-pendulum from the view point of discrete mechanics. First, we show the discrete-time model of the cart-pendulum by using discrete mechanics. Next, we derive a stabilization control method for the continuous-time cart-pendulum based on discrete mechanics, and then numerical simulations are shown. Finally, we apply the proposed method to the actual cart-pendulum to show the validity of our method.

1. Introduction

In recent years, *Discrete mechanics* has been focused on and attracted a lot of attention as a new discretizing technique for mechanical systems [1, 2, 3]. It is known that discrete mechanics has some interesting properties: (i) it can describe energies for conservative/dissipative systems with less errors, (ii) some laws of physics such as Noether's theorem are satisfied. (iii) simulations can be performed for large sampling times. Therefore, we can expect that discrete mechanics is available for designing controllers with a high affinity for computers. However, there exist few researches on control of mechanical systems via discrete mechanics [4, 5, 8, 9].

In [9], we have proposed a control strategy for the continuous-time cart-pendulum with friction, and a transformation method to zero-order hold inputs. However, any experiments by using discrete mechanics have not been done so far, and we have a question about whether discrete mechanics is available for control of actual mechanical systems. The purpose of this paper is to answer the question above, and hence we apply a controller design technique by discrete mechanics to the actual cart-pendulum as a simple mechanical system.

This paper is organized as follows. In Section 2, we derive the discrete model of the cart-pendulum by using discrete mechanics. Next, Section 3 gives a stabilizing control method for the discrete cart-pendulum based on discrete-time optimal regulator theory, and a transformation method from discrete-time inputs to zero-order hold inputs by discrete mechanics is porposed. Simulations are also shown to check the effectiveness of the proposed method. Finally, in Section 6 we carry out some experiments of the actual cart-pendulum to show the application potentiality of our method.

2. Discrete Cart-Pendulum

In this section, we derive the discrete-time model of the cart-pendulum as shown in Fig. 1 by discrete mechanics. See [1, 3, 9] for details of discrete mechanics. Let $\theta \in \mathbf{S} := (-\pi, \pi]$ be the angle of the pendulum and $z \in \mathbf{R}$ be the position of the cart. We set parameters of the system as follows: m : the mass of the pendulum, M : the mass of the cart, l : the length of the pendulum, η, μ : friction coefficients of the pendulum and the cart, respectively. The Lagrangian of this system is given by

$$L = \frac{1}{2}ml^2\dot{\theta}^2 + ml\dot{\theta}\dot{z}\cos\theta + \frac{1}{2}(m+M)\dot{z}^2 - mgl\cos\theta, \quad (1)$$

and we then have the discrete Lagrangian with the discrete variables shown in Fig. 1:

$$\begin{aligned} L^d =& \frac{m+M}{2h}(z_{k+1}-z_k)^2 + \frac{ml^2}{2h}(\theta_{k+1}-\theta_k)^2 \\ &+ \frac{ml}{h}\cos\{(1-\alpha)\theta_k + \alpha\theta_{k+1}\}(z_{k+1}-z_k)(\theta_{k+1}-\theta_k) \\ &- mglh\cos\{(1-\alpha)\theta_k + \alpha\theta_{k+1}\}. \end{aligned} \quad (2)$$

Consequently, from (2), we derive the discrete Euler-Lagrange equation of the cart-pendulum with friction as (3) and (4).

Substituting $\theta_{k-1} = \theta_k = \theta_{k+1}$, $z_{k-1} = z_k = z_{k+1}$ and $u_k = 0$ into (3) and (4), we have $\sin\theta_k = 0$. Therefore, the equilibria of the discrete cart-pendulum are $(\theta_k, z_k) = (0, z^e)$, (π, z^e), $\forall z^e \in \mathbf{R}$, that is, they correspond with those of the usual cart-pendulum in the continuous setting. Finally, we calculate the linear approximation system that behaves around the equilibrium $\theta_k = 0$. Considering $\theta_{k-1}, \theta_k, \theta_{k+1} \approx 0$ for (3) and (4), we obtain the linear approximation as (5) and (6).

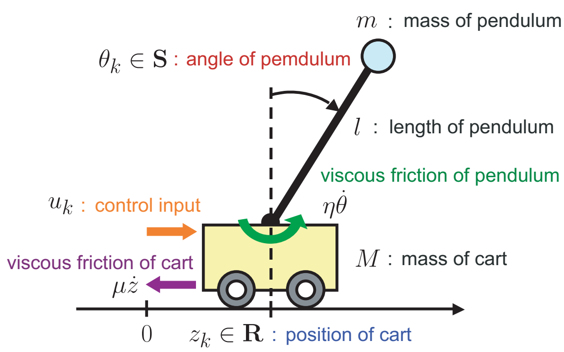

Fig. 1 : Cart-Pendulum System

$$
\begin{aligned}
&-ml(1-\alpha)(\theta_{k+1}-\theta_k)(z_{k+1}-z_k)\sin\{(1-\alpha)\theta_k+\alpha\theta_{k+1}\}-ml\cos\{(1-\alpha)\theta_k+\alpha\theta_{k+1}\}(z_{k+1}-z_k)-ml^2(\theta_{k+1}-\theta_k)\\
&\quad+mgl(1-\alpha)h^2\sin\{(1-\alpha)\theta_k+\alpha\theta_{k+1}\}+ml\cos\{(1-\alpha)\theta_{k-1}+\alpha\theta_k\}(z_k-z_{k-1})+ml^2(\theta_k-\theta_{k-1})\\
&\qquad-ml\alpha(\theta_k-\theta_{k-1})(z_k-z_{k-1})\sin\{(1-\alpha)\theta_{k-1}+\alpha\theta_k\}+mgl\alpha h^2\sin\{(1-\alpha)\theta_{k-1}+\alpha\theta_k\}\\
&\qquad\quad+\eta\{(1-\alpha)(\theta_{k+1}-\theta_k)+\alpha(\theta_k-\theta_{k-1})\}=0
\end{aligned}
\tag{3}
$$

$$
\begin{aligned}
&-(m+M)(z_{k+1}-z_k)-ml(\theta_{k+1}-\theta_k)\cos\{(1-\alpha)\theta_k+\alpha\theta_{k+1}\}+(m+M)(z_k-z_{k-1})\\
&\quad+ml(\theta_k-\theta_{k-1})\cos\{(1-\alpha)\theta_k+\alpha\theta_{k-1}\}+\mu\{(1-\alpha)(z_{k+1}-z_k)+\alpha(z_k-z_{k-1})\}+hu_k=0
\end{aligned}
\tag{4}
$$

$$
\begin{aligned}
&-ml(z_{k+1}-z_k)+mgl(1-\alpha)h^2\{(1-\alpha)\theta_k+\alpha\theta_{k+1}\}+ml(z_k-z_{k-1})-ml^2(\theta_{k+1}-\theta_k)+ml^2(\theta_k-\theta_{k-1})\\
&\quad+mgl\alpha h^2\{(1-\alpha)\theta_{k-1}+\alpha\theta_k\}+\eta\{(1-\alpha)(\theta_{k+1}-\theta_k)+\alpha(\theta_k-\theta_{k-1})\}=0
\end{aligned}
\tag{5}
$$

$$
\begin{aligned}
&-(m+M)(z_{k+1}-z_k)-ml(\theta_{k+1}-\theta_k)+ml(\theta_k-\theta_{k-1})+(m+M)(z_k-z_{k-1})\\
&\quad+\mu\{(1-\alpha)(z_{k+1}-z_k)+\alpha(z_k-z_{k-1})\}+hu_k=0
\end{aligned}
\tag{6}
$$

3. Controller Design base on Discrete Mechanics

3.1. Proposed Control Method

This subsection presents a stabilizing controller for the continuous-time cart-pendulum based on discrete mechanics. First, we sum up a stabilizing method for the discrete cart-pendulum derive in [8, 9]. We set a state variable as $x_k = [\,x_k^1\ \ x_k^2\ \ x_k^3\ \ x_k^4\,]^{\mathsf{T}} = [\,\theta_{k-1}\ \ \theta_k\ \ z_{k-1}\ \ z_k\,]^{\mathsf{T}}$. From (5) and (6), we obtain the discrete-time linear control system:

$$x_{k+1} = Ax_k + Bu_k, \tag{7}$$

where $A \in \mathbf{R}^{4\times4}$, $B \in \mathbf{R}^{4\times1}$ are appropriate matrices. By using the discrete-time optimal regulator theory for (7), we can have a stabilizing controller in the form:

$$u_k = Kx_k \tag{8}$$

where $K = [\,K_1\ \ K_2\ \ K_3\ \ K_4\,]^{\mathsf{T}} \in \mathbf{R}^{1\times4}$ is a gain matrix. In [9], we showed that the control method can be stabilize the discrete cart-pendulum at not only a small sampling time but also a larger one.

Next, we give a transformation method from a discrete-time controller into a continuous-time controller. We consider a zero-order hold input, which is one of simplest continuous-time controllers, as follows:

$$u^c(t) = Lx_k,\ \ kh \le t < (k+1)h, \tag{9}$$

where $L = [L_1\ L_2\ L_3\ L_4]^{\mathsf{T}} \in \mathbf{R}^{1\times4}$ is a gain matrix. That is, (9) implies a state feedback law using the value of x_k during $kh \le t < (k+1)h$. From discrete-time Lagrange d'Alembert principle [3], the following theorem can be derived.

Theorem 1: Assume that α is sufficiently small. By discrete Lagrange-d'Alembert principle, the gain matrix of the zero-order hold input is approximately obtained as

$$
\begin{aligned}
L_1 &= \frac{K_1}{(1-\alpha)h} - \frac{\alpha K_2}{(1-\alpha)^2 h}, \quad L_2 = \frac{K_2}{(1-\alpha)h}\\
L_3 &= \frac{K_3}{(1-\alpha)h} - \frac{\alpha K_4}{(1-\alpha)^2 h}, \quad L_4 = \frac{K_4}{(1-\alpha)h}
\end{aligned}
\tag{10}
$$

from the discrete input (8). ∎

In Theorem 1, we impose the assumption on not h but α, so the controller can be available for a larger sampling time. Therefore, we can expect that the controller utilizes the advantages of discrete mechanics.

3.2. Numerical Simulation

In this subsection, we show some simulations on stabilization of the continuous-time cart-pendulum. We use parameters as follows: $m = 0.05$ [kg], $M = 0.55$ [kg], $l = 0.21$ [m], $\eta = 1.14 \times 10^{-4}$ [Nms/rad], $\mu = 2.477$ [Ns/m], $\alpha = 0.001$, and the initial condition: $\theta = 0.1$ [rad], $\dot{\theta} = 0$ [rad/s], $z = 0.05$ [m], $\dot{z} = 0$ [rad/s]. We consider two kinds of sampling times: $h = 0.05$ [s] and $h = 0.1$ [s]. Since the period of the pendulum is $T = 1.07$ [s], the sampling time $h = 0.1$ [s] seems to be large.

Fig. 2 and 4 show the zero-order hold input derived by (9) and (10) at the sampling time $h = 0.05$ and $h = 0.1$, respectively. Fig. 3 and 5 depict the time responses of θ and z at the sampling time $h = 0.05$ and $h = 0.1$, respectively. From these figures, it can be confirmed that the continuous-time cart-pendulum is stabilized by the proposed zero-order hold input at not only a small sampling $h = 0.05$ [s] time but also a larger one $h = 0.1$ [s]. Therefore, we can say that our proposed method is available for stabilization of the continuous-time cart-pendulum.

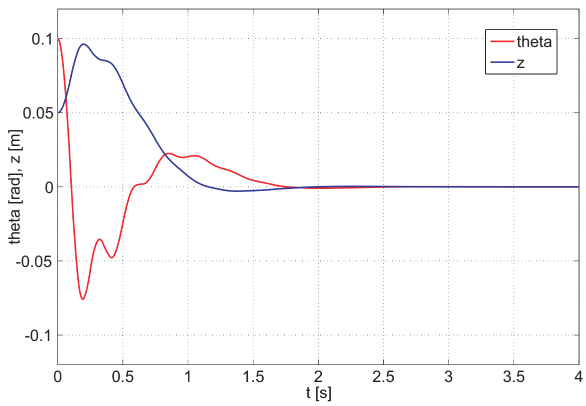

Fig. 2 : Time Series of θ and z (Simulation, $h = 0.05$)

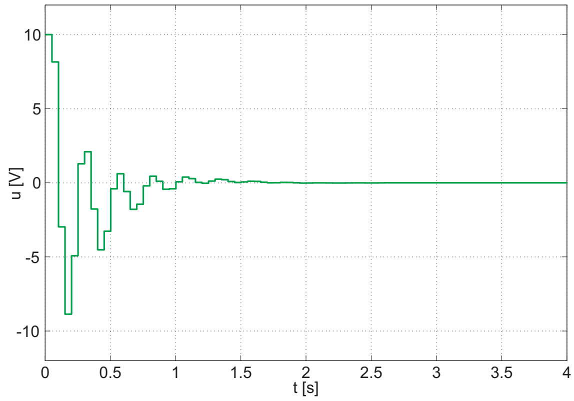

Fig. 3 : Time Series of u (Simulation, h = 0.05)

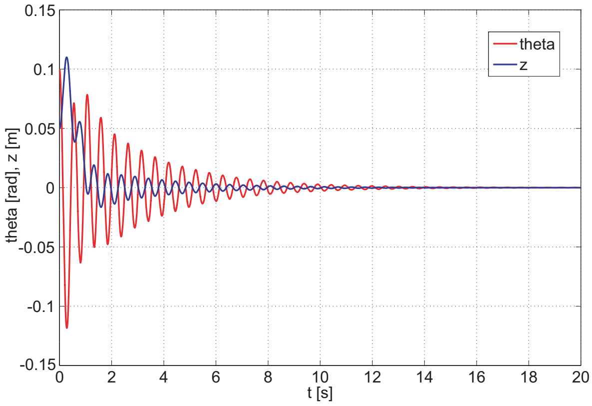

Fig. 4 : Time Series of θ and z (Simulation, h = 0.1)

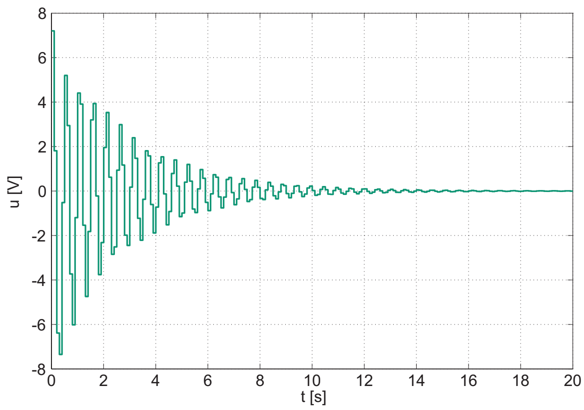

Fig. 5 : Time Series of u (Simulation, h = 0.1)

4. Experiment

4.1. Problem Setting

In this section, we apply the control method shown in the previous section to the actual cart-pendulum in order to examine the application potentiality of discrete mechanics. The cart-pendulum equipment is shown in Fig. 6 and its parameters are m = 0.05 [kg], M = 0.55 [kg], l = 0.21 [m], $\eta = 1.14 \times 10^{-4}$ [Nms/rad], μ = 2.477 [Ns/m], which are the same values as the ones used in the simulations.

We consider two kinds of sampling times: h = 0.05 [s] and h = 0.1 [s] and use α = 0.001, which are also same values as the ones used in the simulations. The initial condition of the cart-pendulum is as follows: the angle of the pendulum: 0.1 [rad], the angular velocity of the pendulum: 0 [rad/s] the position of the cart: 0.05 [m], the angular velocity of the pendulum: 0 [m/s].

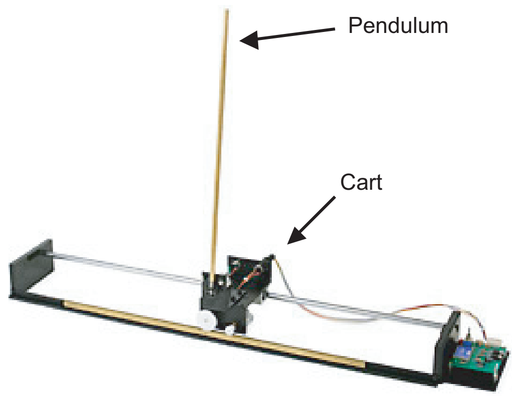

Fig. 6 : Laboratory Equipment of Cart-Pendulum

4.2. Experimental Results

Based on the problem setting shown in the previous subsection, we perform some experiments of the actual cart-pendulum.

Fig. 7–10 illustrate the results of the experiments. Fig. 7 and 9 show the zero-order hold input derived by (9) and (10) at the sampling time h = 0.05 and h = 0.1, respectively. Fig. 8 and 10 depict the time responses of the pendulum and the cart at the sampling time h = 0.05 and h = 0.1, respectively. From these results, we can confirm that the cart-pendulum is stabilized by the proposed zero-order hold input, and for h = 0.1 [s], which is a larger sampling time in comparison with the pendulum's period, the cart-pendulum is stabilized. Hence, we can say that our control method is also available for stabilization of the actual cart-pendulum.

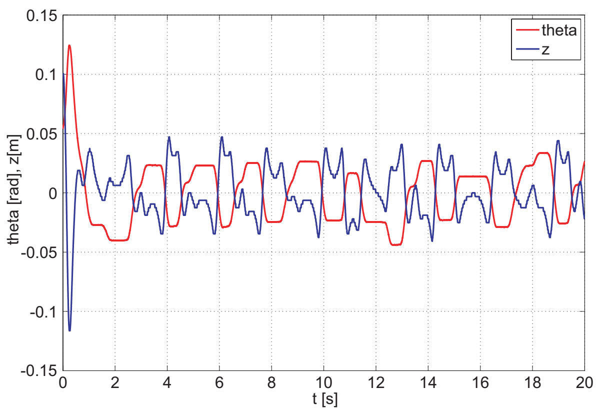

Fig. 7 : Time Series of θ and z (Experiment, h = 0.05)

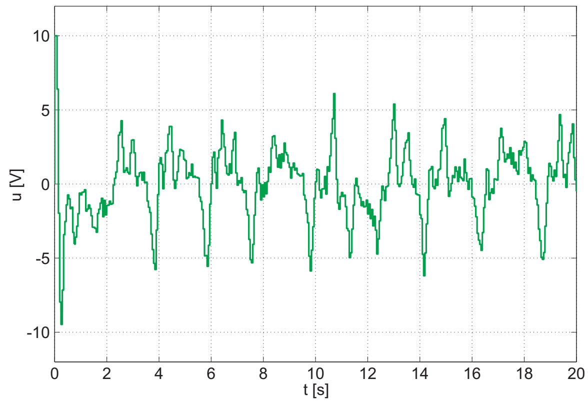

Fig. 8 : Time Series of u (Experiment, h = 0.05)

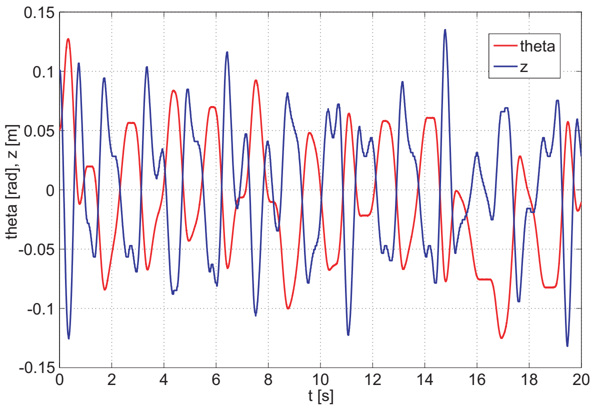

Fig. 9 : Time Series of θ and z (Experiment, h = 0.1)

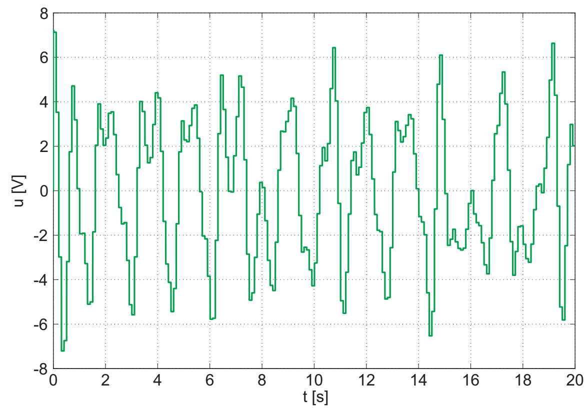

Fig. 10 : Time Series of u (Experiment, h = 0.1)

5. Conclusion

In this paper, we have presented a stabilizing control method for the cart-pendulum and a transformation method from discrete-time inputs to continuous-time zero-order hold inputs from the viewpoint of discrete mechanics. In both simulations and experiments, we have shown that the proposed method can stabilize not only the continuous-time cart-pendulum but also the actual cart-pendulum.

Our future work is as follows: a stabilizing control method based on model predictive control swing-up control of the discrete cart-pendulum, applications to other mechanical systems such as gaits of humanoid robots.

References

[1] J. E. Marsden, G. W. Patrick and S. Shkoller, "Multisymplectic Geometry, Variational Integrators and Nonlinear PDEs," *Comm. in Math. Phys.*, vol.199, pp.351–395, 1998.

[2] C. Kane, J. E. Marsden, M. Ortiz and M. West, "Variational Integrators and the Newmark Algorithm for Conservative and Dissipative Mechanical Systems," *Int. J. for Numer. Meth. in Engineering*, vol.49, pp.1295–1325, 2000.

[3] J. E. Marsden and M. West, "Discrete Mechanics and Variational Integrators," *Acta Numerica*, vol.10, pp.3571–5145, 2001.

[4] A. M. Bloch, M. Leok, J. E. Marsden and D. V. Zenkov, "Controlled Lagrangians and Stabilization of the Discrete Cart-Pendulum System," *in Proc. of 44th IEEE CDC-ECC*, Seville, Spain, pp.6579–6584, 2005.

[5] A. M. Bloch, M. Leok, J. E. Marsden and D. V. Zenkov, "Controlled Lagrangians and Potential Shaping for Stabilization of the Discrete Mechanical Systems," *in Proc. of 45th IEEE CDC*, San Diego, USA, pp.3333–3338, 2006.

[6] D. G. Luenberger, "Non-linear Descriptor Systems," *J. Econom. Dynam. Cont.*, vol.1, pp.219–242, 1979.

[7] T. Fliegner, Ü. Kotta and H. Nijmeijer, "Solvability and Right-inversion of Implicit Nonlinear Discrete-time Systems," *SIAM J. Cont. and Optim.*, vol.34, no.6, pp.2092–2115, 1996.

[8] T. Kai and Y. Yamamoto, "On Analysis and Control of the Cart-Pendulum System Modeled by Discrete Mechanics," *in Proc. of NOLTA 2007*, Vancouver, Canada, pp.485–488, 2007.

[9] T. Kai and K. Bito, "Solvability Analysis and Stabilization of the Cart-Pendulum Modeled by Discrete Mechanics with Friction," *in Proc. of NOLTA 2008*, Budapest, Hungary, pp.305-308, 2008.