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Computer assisted proofs for solutions to nonlinear elliptic partial differential equations on arbitrary polygonal domain

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Abstract—In this article, a computer assisted proof framework is introduced. Computer assisted proof method shows the existence and local uniqueness of exact solutions of nonlinear PDEs. Based on Newton-Kantorovich theorem, our numerical method is another variant of computer assisted proofs which provides verified numerical error estimates by numerical computations with result verification. One feature is that, by adopting hyper-circle equation in a posteriori error estimate, our method can deal with arbitrary polygonal domain. Furthermore, Raviart-Thomas mixed finite element enables us to get an effective residual bounds with respect to an operator equation. Our approach is presented first for an abstract problem. Then, some numerical results are demonstrated.

1. CAP framework

Let us explain our computer assisted approach first for the following abstract problem:

$$\text{Find } u \in V \text{ satisfying } \mathcal{F}(u) = 0, \tag{1}$$

with V denoting a Hilbert space with its inner product $(\cdot,\cdot)_V$. We also define the dual space of V as V^*. Let $\mathcal{F} : V \to V^*$ denote some Fréchet differentiable mapping. Let $\hat{u} \in V$ be an approximate solution to (1), and Fréchet derivative of $\mathcal{F}$ at $\hat{u}$ denotes $\mathcal{F}'[\hat{u}] : V \to V^*$, *i.e.* satisfying

$$\|\mathcal{F}(\hat{u}+v) - \mathcal{F}(\hat{u}) - \mathcal{F}'[\hat{u}]v\|_{V^*} = o(\|v\|_V),\ \|v\|_V \to 0.$$

Assuming that we know three constants K, δ_h and L_c such that

$$\|\mathcal{F}'[\hat{u}]^{-1}\|_{V^*,V} \le K, \tag{2}$$

i.e., K bounds the inverse operator of $\mathcal{F}'[\hat{u}]$. δ_h bounds the residual of approximation:

$$\|\mathcal{F}(\hat{u})\|_{V^*} \le \delta_h. \tag{3}$$

L_c denotes the Lipschitz constant of $\mathcal{F}'$, which is required to be Lipschitz continuous on the certain ball $D \subset V$,

$$\|\mathcal{F}'[v] - \mathcal{F}'[w]\|_{V,V^*} \le L_c\|v - w\|_V, \quad \forall v, w \in D. \tag{4}$$

Our main task to computer assisted analysis is the calculation of these constants explicitly. In order to prove the existence and local uniqueness of the exact solution in the neighborhood of $\hat{u}$, the following theorem is applicable to (1). This theorem is called Newton-Kantorovich theorem [1]. After that we give an elementary proof based on Banach's fixed point theorem.

Theorem 1 *Assuming that the Fréchet derivative $\mathcal{F}'[\hat{u}]$ is nonsingular and satisfies*

$$\|\mathcal{F}'[\hat{u}]^{-1}\mathcal{F}(\hat{u})\|_V \le \alpha,$$

for a certain positive α. Then, let $\overline{B}(\hat{u}, 2\alpha) := \{v \in V : \|v - \hat{u}\|_V \le 2\alpha\}$ be a closed ball centered at $\hat{u}$ with radius 2α. Let also $D \supset \overline{B}(\hat{u}, 2\alpha)$ be an open ball on V. We assume that for a certain positive ω, the following holds:

$$\|\mathcal{F}'[\hat{u}]^{-1}(\mathcal{F}'[v] - \mathcal{F}'[w])\|_{V,V} \le \omega\|v - w\|_V, \quad \forall v, w \in D.$$

If $\alpha\omega \le \frac{1}{2}$ holds, then there is a solution $u \in V$ of (1) satisfying

$$\|u - \hat{u}\|_V \le \rho := \frac{1 - \sqrt{1 - 2\alpha\omega}}{\omega}. \tag{5}$$

Furthermore, the solution u is unique in (5).

Since $\alpha \le K\delta_h$ and $\omega \le KL_c$ form (2)-(4), the concrete computation of K, δ_h and L_c yields computer assisted proof of the existence and local uniqueness to the problem (1). Therefore, if

$$\alpha\omega \le K^2\delta_h L_c \le 1/2$$

is obtained by verified computations, then the existence and local uniqueness of the solution are proved numerically.

Remark 1 *Our computer assisted proof method requires the approximate solution of (1) in a certain finite dimensional subspace, such as the finite element subspace of V. It means that we can verify the solution when one have the approximate solution of (1) in the discrete subspace of V.*

2. Variational form

Let Ω be a bounded polygonal domain in $\mathbb{R}^2$ with arbitrary shape. In this article, we are concerned with Dirichlet boundary value problem of the semilinear elliptic equation of the form:

$$\begin{cases} -\Delta u = f(u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{6}$$

where $f : V \to X$ is assumed to be Fréchet differentiable. For example, the following function

$$f(u) = -b \cdot \nabla u - cu + c_2 u^2 + c_3 u^3 + g$$

with $b(x) \in (L^\infty(\Omega))^m$, $c, c_2, c_3 \in L^\infty(\Omega)$ and $g \in X$ satisfies this condition.

In the classical analysis of variational theory, the solution to the Dirichlet boundary problem (6) satisfies the variational boudary value problem: Find $u \in V$ such that

$$(\nabla u, \nabla v) = (f(u), v), \quad \text{for all } v \in V. \tag{7}$$

Here,

$$(\nabla u, \nabla v) := \int_\Omega \nabla u \cdot \nabla v dx, \quad \text{and} \quad (f(u), v) := \int_\Omega f(u) v dx.$$

The original problem (6) is transformed into (7) equivalently. Further it is represented by

$$\mathcal{A}u = \mathcal{N}(u), \tag{8}$$

where the operator $\mathcal{A} : V \to V^*$ is defined by

$$\langle \mathcal{A}u, v \rangle := A(u, v) = (\nabla u, \nabla v), \quad \text{for all } v \in V$$

and $\mathcal{N} : V \to V^*$ is denoted by

$$\langle \mathcal{N}(u), v \rangle = (f(u), v), \quad \text{for all } v \in V.$$

Therefore, we define the operator $\mathcal{F} : V \to V^*$ by $\mathcal{F}(u) := \mathcal{A}u - \mathcal{N}(u)$. Eq.(8) can be written as

$$\mathcal{F}(u) = 0.$$

This is nothing but the abstract problem (1).

In order to apply Newton-Kantorovich theorem, the Fréchet derivative of $\mathcal{F}$ is needed. The Fréchet differentiability of $\mathcal{F}$ is derived by that of f. We can define an operator $\mathcal{N}'[\hat{u}] : V \to V^*$ by

$$\langle \mathcal{N}'[\hat{u}]u, v \rangle := (f'(\hat{u})u, v), \quad \forall v \in V. \tag{9}$$

For a given $\hat{u} \in V$, the Fréchet derivative $\mathcal{F}'[\hat{u}] : V \to V^*$ of $\mathcal{F} : V \to V^*$ at $\hat{u}$ is given as

$$\mathcal{F}'[\hat{u}] = \mathcal{A} - \mathcal{N}'[\hat{u}].$$

Now, we define the natural embedding operator $i_{(X \hookrightarrow V^*)} : X \to V^*$. For fixed $w \in X$, we can define

$$\langle i_{(X \hookrightarrow V^*)} w, v \rangle := (w, v) \quad \text{for all } v \in V.$$

Since $i_{(X \hookrightarrow V^*)} : X \to V^*$ is compact and $f'(\hat{u}) : V \to X$ is continuous, the composite operator

$$\mathcal{N}'[\hat{u}] = i_{(X \hookrightarrow V^*)} \circ f'(\hat{u}) : V \to V^*$$

is compact.

3. Properties for arbitrary polygonal domain

Let V be Hilbert space and V_h be its finite approximation. For the computer assistance, we need an error estimate of an orthogonal projection $\mathcal{P}_h^1 : V \to V_h$, which is defined by

$$A(u - \mathcal{P}_h^1 u, v_h) = 0, \quad \forall v_h \in V_h.$$

The error estimate is given by the following theorem. Espicially, we emphasize this estimation works on non-convex domain by adopting hyper-circle equation.

Theorem 2 (Liu and Oishi [2]) *For $f \in L^2(\Omega)$, let $u \in V$ and $\mathcal{P}_h^1 u \in V_h$ be solutions of*

$$(\nabla u, \nabla v) = (f, v), \quad \forall v \in V$$

and

$$\left(\nabla(\mathcal{P}_h^1 u), \nabla v_h\right) = (f, v_h), \quad \forall v_h \in V_h,$$

respectively. Putting $C_M := \sqrt{(C_h^0)^2 + \kappa^2}$, a posteriori estimation is obtained

$$|u - \mathcal{P}_h^1 u|_{H^1} \le C_M \|f\|_{L^2},$$

$$\|u - \mathcal{P}_h^1 u\|_{L^2} \le C_M |u - \mathcal{P}_h^1 u|_{H^1} \le (C_M)^2 \|f\|_{L^2}.$$

Here, C_h^0 is error estimate of another orthogonal projection, which maps $L^2(\Omega)$ to piecewise constant functions (denoting M_h) defined by

$$(u - \mathcal{P}_h^0 u, v_h) = 0, \quad \forall u \in L^2(\Omega),\ v_h \in M_h.$$

Furthermore, κ is a computable quantity such that

$$\kappa := \max_{0 \ne f_h \in M_h} \min_{v_h \in V_h} \min_{p_h \in W_{f_h}} \frac{\|p_h - \nabla v_h\|_{L^2}}{\|f_h\|_{L^2}},$$

where W_{f_h} is subspace of $H(\mathrm{div}, \Omega)$ satisfying $\mathrm{div}\, p_h + f_h = 0, \forall f_h \in M_h$. Next, Sobolev's embedding constant is calculated as below.

Lemma 1 *Let $\sigma \in [0, \infty)$ denote the minimal point of the spectrum of $-\Delta$ on V. Let $p \in [2, \infty)$ and ν denote the largest integer less than $p/2$. We have*

$$C_{e,p} := \left(\frac{1}{2}\right)^{\frac{1}{2} + \frac{2\nu - 3}{p}} \left[\frac{p}{2}\left(\frac{p}{2} - 1\right) \cdots \left(\frac{p}{2} - \nu + 2\right)\right]^{\frac{2}{p}} \sigma^{-\frac{1}{p}},$$

where the bracket term is put equal to 1 if $\nu = 1$.

Lower bound of the minimal spectrum with respect to $-\Delta$ is given as follows.

Theorem 3 (Liu and Oishi [2]) *Let λ_k be spectrums of $-\Delta$. $\tilde{\lambda}_k$ is assumed to be its discretized approximation with verified computations. C_M is the same as above. Suppose*

$$1 - (C_M)^2 \lambda_k > 0,$$

then each spectrum of $-\Delta$ is bounded by

$$\frac{\tilde{\lambda}_k}{1 + (C_M)^2 \tilde{\lambda}_k} \le \lambda_k \le \tilde{\lambda}_k.$$

4. Each quantity K, δ_h & L_c

For the norm of inverse operator, K is bounded by the following theorem. This theorem is a modification of the main theorem in M.T. Nakao et al. [3] in 2005. Further, another evaluation of K has been computed by M. Plum [4]. Let V_h be a finite element approximation of V and $V_c := V \setminus V_h$ be its orthogonal complement.

Theorem 4 *Let $\mathcal{N}'[\hat{u}] : V \to V^*$ be the linear compact operator defined in (9). For three constants K_1, K_2 and K', we assume*

$$\|f'(\hat{u})u\|_{L^2} \le K_1\|u\|_V, \quad \forall u \in V,$$

$$\|f'(\hat{u})u_c\|_{L^2} \le K_2\|u_c\|_V, \quad \forall u_c \in V_c$$

and

$$\|\mathcal{P}_h^1\mathcal{A}^{-1}\mathcal{N}'[\hat{u}]u_c\|_V \le K'\|u_c\|_V, \quad \forall u_c \in V_c.$$

Assuming that the finite dimensional operator $\mathcal{P}_h^1(\mathcal{I} - \mathcal{A}^{-1}\mathcal{N}'[\hat{u}])|_{V_h} : V_h \to V_h$ is invertible with

$$\left\|\left(\mathcal{P}_h^1(\mathcal{I} - \mathcal{A}^{-1}\mathcal{N}'[\hat{u}])|_{V_h}\right)^{-1}\right\|_{V,V} \le \tau.$$

Here, $\mathcal{P}_h^1(\mathcal{I} - \mathcal{A}^{-1}\mathcal{N}'[\hat{u}])|_{V_h} : V_h \to V_h$ is the restriction of $\mathcal{P}_h^1(\mathcal{I} - \mathcal{A}^{-1}\mathcal{N}'[\hat{u}]) : V \to V_h$ to V_h. Moreover, the error estimate of $\mathcal{P}_h^1$ is obtained for given $f \in L^2(\Omega)$:

$$\|u - \mathcal{P}_h u\|_V \le C_M\|f\|_{L^2}.$$

If $C_M(K_1\tau K' + K_2) < 1$, then $(\mathcal{A} - \mathcal{N}'[\hat{u}]) : V \to V^*$ is invertible and enjoys

$$\|(\mathcal{A} - \mathcal{N}'[\hat{u}])^{-1}\|_{V^*,V} \le \sqrt{R^2 + S^2} =: K.$$

Here,

$$R := \frac{\sqrt{(C_M K_1\tau)^2 + 1}}{1 - C_M(K_1\tau K' + K_2)} \quad \text{and} \quad S := \tau(K'R + 1).$$

The defect bound of residual is bounded by computer assistance. Raviart-Thomas mixed finite elements yields a general smoothing technique.

$$\begin{aligned}
\|\mathcal{F}\hat{u}\|_{V^*} &= \sup_{0\neq v\in V}\frac{|A(\hat{u}, v) - (f(\hat{u}), v)|}{\|v\|_V} \\
&= \sup_{0\neq v\in V}\frac{|(\nabla\hat{u}, \nabla v) - (f(\hat{u}), v)|}{\|v\|_V} \\
&= \sup_{0\neq v\in V}\frac{|(\nabla\hat{u} - p_h, \nabla v) + (p_h, \nabla v) - (f(\hat{u}), v)|}{\|v\|_V} \\
&\le \sup_{0\neq v\in V}\frac{|(\nabla\hat{u} - p_h, \nabla v)|}{\|v\|_V} + \sup_{0\neq v\in V}\frac{|(\text{div}\, p_h + f(\hat{u}), v)|}{\|v\|_V} \\
&\le \|\nabla\hat{u} - p_h\|_{L^2} + \sup_{0\neq v\in V}\frac{|(f(\hat{u}) - f_h(\hat{u}), v - \mathcal{P}_n^0 v)|}{\|v\|_V} \\
&\le \|\nabla\hat{u} - \tilde{p}_h\|_{L^2} + C_h^0\,\|f(\hat{u}) - f_h(\hat{u})\|_{L^2} =: \delta_h,
\end{aligned}$$

where $\tilde{p}_h$ is an interval function which contains the exact function *i.e.* $p_h \in W_{f_h}$. The interval obtained by verified computations and $f_h \in M_h$.

Finally, we estimate Lipschitz constant of $\mathcal{F}'[u]$. Here, we assume that $f' : V \to \mathcal{L}(V, L^2(\Omega))$ is Lipschitz continuous on D. Namely, there exists a positive constant C_L satisfying

$$\left|((f'(v) - f'(w))u, \psi)\right| \le C_L\|v - w\|_V\|u\|_V\|\psi\|_V$$

for $v, w \in D$ and $u, \psi \in V$. Usually, the optimal estimation depends on the definition of f. For $v, w \in D$, we have

$$\begin{aligned}
&\|\mathcal{F}'[v] - \mathcal{F}'[w]\|_{V,V^*} \\
= &\sup_{u\in V\setminus\{0\}}\sup_{\psi\in V\setminus\{0\}}\frac{|\langle(\mathcal{N}'[v] - \mathcal{N}'[w])u, \psi\rangle|}{\|u\|_V\|\psi\|_V} \\
= &\sup_{u\in V\setminus\{0\}}\sup_{\psi\in V\setminus\{0\}}\frac{|((f'(v) - f'(w))u, \psi)|}{\|u\|_V\|\psi\|_V} \\
\le &\; C_L\|v - w\|_V.
\end{aligned}$$

Therefore, one can put $L_c := C_L$. Three quantities are computable.

5. Computational results

Now, we shall present a numerical result. All computations are carried out on Mac OS X, 2.26GHz Quad-Core Intel Xeon by using MATLAB 2010a with a toolbox for verified computations, INTLAB [5]. We also use the mesh generator Gmsh [6]. For an application of our computer assisted proof method, we treat a nonlinear Dirichlet boundary value problem on several polygonal domain:

$$\begin{cases} -\Delta u = u^2 + \eta, & \text{in } \Omega, \\ u(x) = 0, & \text{on } \partial\Omega, \end{cases} \tag{10}$$

where $\eta \in \mathbb{R}$. Obviously, the Fréchet derivative of right-hand side is given by $f'(\hat{u}) = 2\hat{u}$. An approximate solution $\hat{u}$ is calculated by FEM. The approximate solution is bounded on Ω then $\hat{u}$ is the element of $L^\infty(\Omega)$ in this solution. So that for $\hat{u} \in L^\infty(\Omega) \cap V$, we have

$$\|f'(\hat{u})u\|_{L^2} \le 2C_{e,2}\|\hat{u}\|_{L^\infty}\|u\|_V, \quad \forall u \in V,$$

$$\|f'(\hat{u})u_c\|_{L^2} \le 2C_h^1\|\hat{u}_c\|_{L^\infty}\|u_c\|_V, \quad \forall u_c \in V_c$$

and Lipschitz continuity of $\mathcal{F}'[u]$. For $u \in V$ and $v, w \in D$, we have

$$\left|((f'(v) - f'(w))u, \psi)\right| \le 2C_{e,3}^3\|\hat{u}\|_{L^\infty}\|v - w\|_V\|u\|_V\|\psi\|_V$$

Rectangular domain

We consider the following equation on Rectangle domain

$$\begin{cases} -\Delta u = u^2 + 10, & \text{in } \Omega, \\ u(x) = 0, & \text{on } \partial\Omega. \end{cases}$$

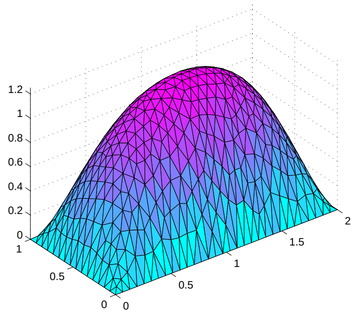

Fig.1: Apptroximate solution on $(0, 1) \times (0, 2)$

The following results are obtained by our computer approach. Here, h denotes mesh size. σ : lower bound of minimal spectrum $-\Delta$, K : norm of inverse operator, δ_h : residual bound and ρ : error estimate which contain unique solution.

Table 1: Computational results

h	σ	K	δ_h	ρ
2^{-3}	11.649	1.351	1.183	1.891
2^{-4}	12.206	1.285	5.725×10^{-1}	7.805×10^{-1}
2^{-5}	12.299	1.264	2.974×10^{-1}	3.867×10^{-1}
2^{-6}	12.328	1.253	1.478×10^{-1}	1.878×10^{-1}
2^{-7}	12.335	1.249	7.528×10^{-2}	9.459×10^{-2}

Hexagonal domain

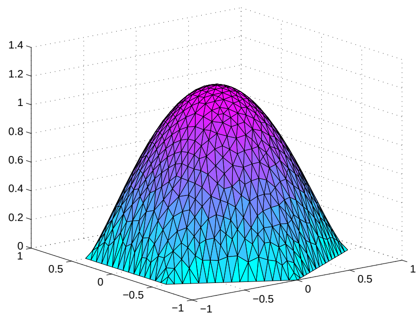

Fig.2: Apptroximate solution on hexagonal domain where $\eta = 5$ in (10)

Table 2: Computational results

h	σ	K	δ_h	ρ
2^{-3}	7.013	1.741	6.344×10^{-1}	1.498
2^{-4}	7.114	1.709	3.373×10^{-1}	6.479×10^{-1}
2^{-5}	7.144	1.693	1.785×10^{-1}	3.194×10^{-1}
2^{-6}	7.151	1.686	9.614×10^{-2}	1.667×10^{-1}
2^{-7}	7.154	1.681	4.680×10^{-2}	7.972×10^{-2}

L-shape domain

Our evaluation is applied to non-convex domain. For a model problem, we consider in case of $\eta = 10$ in (10).

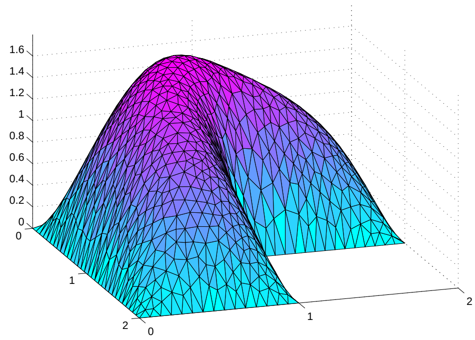

Fig.3: Apptroximate solution on $(0, 2) \setminus (1, 2)$

Table 3: Computational results

h	σ	K	δ_h	ρ
2^{-3}	7.013	1.741	6.344×10^{-1}	1.498
2^{-4}	7.114	1.709	3.373×10^{-1}	6.479×10^{-1}
2^{-5}	7.144	1.693	1.785×10^{-1}	3.194×10^{-1}
2^{-6}	7.151	1.686	9.614×10^{-2}	1.667×10^{-1}
2^{-7}	7.154	1.681	4.680×10^{-2}	7.972×10^{-2}

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