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Recurrence-based evolving networks for time series analysis of complex systems

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Abstract—This paper presents a novel approach for analyzing the structural properties of time series from real-world complex systems by means of evolving complex networks. Starting from the concept of recurrences in phase space, the recurrence matrices corresponding to different parts of a time series are re-interpreted as the adjacency matrices of complex networks, which link different observations if the associated temporal evolution is sufficiently similar. We provide some illustrative examples demonstrating that the local properties of the resulting *recurrence networks* allow identifying dynamically invariant objects in the phase space of complex systems. Moreover, changes in the global network properties of evolving recurrence networks allow identifying time intervals containing hidden dynamical transitions, which is exemplified for some financial time series.

1. Introduction

During the last years, classical graph theory has been systematically extended and applied for studying real-world networks in various scientific disciplines. In particular, the corresponding results have triggered substantial progress in our understanding of the interplay between structure and dynamics of complex networks, *i.e.,* systems that are composed of a number of mutually interacting units [1, 2, 3, 4].

In 2006, Zhang and Small [5, 6, 7] suggested studying the topological features of pseudo-periodic time series in terms of complex networks. For this purpose, individual cycles (defined by distinct minima or maxima of the studied time series) have been considered as vertices of a network, the connectivity of which has been established by different proximity measures. A similar approach applicable also to time series without obvious oscillatory components has been suggested by Yang and Yang [8], considering embedded time series and the resulting phase space vectors as vertices, whose mutual (Pearson) correlation determines the network connectivity. Both approaches are based on the mutual proximity of different parts of a time series in a certain abstract space and utilize thresholds to this proximity for determining a network pattern. A general alternative to such proximity networks has been recently suggested by Lacasa *et al.* [9] in terms of so-called visibility graphs.

In this work, we apply an alternative threshold-based concept, which exploits recurrences in phase space. In this formalism, a state (phase space vector) $X(t_i)$ is said to be recurrent if there is $t_j \neq t_i$ such that $d(X(t_i) - X(t_j)) < \varepsilon$ for some distance measure $d(\cdot,\cdot)$ in phase space. Under general conditions, the structure of recurrences in phase space can be simply encoded in terms of the recurrence matrix [10, 11]

$$R_{i,j} = \Theta(\varepsilon - d(X(t_i) - X(t_j))) \quad (1)$$

where $\Theta(\cdot)$ is the Heaviside function. Following the above considerations, this matrix can be re-interpreted as the adjacency matrix of an unweighted complex network associated with the given time series (more specifically, the adjacency matrix is given by $A_{i,j} = R_{i,j} - \delta_{i,j}$), which is called ($\varepsilon-$) recurrence network [12, 13, 14, 15]. Note that such recurrence networks and closely related concepts have been independently suggested by a variety of authors (see [14, 15] for details).

The advantage of considering the concept of recurrences instead of defining distances in terms of correlations [8] is that it allows generating networks based on individual observations without any embedding or consideration of groups of observations. Recent results have revealed that some of the fundamental dynamical invariants of complex systems are conserved in the recurrence matrices obtained without embedding [16] (even more, embedding is found to sometimes induce spurious correlations [17]). This preservation property allows the full reconstruction of a time series from its recurrence matrix (modulo some rescaling of its probability distribution function) [18, 19].

We have to underline that the properties of a recurrence network do not depend on the temporal order of vertices, *i.e.,* the corresponding complex network measures are invariant under permutations of the observations in the underlying time series. In this sense, recurrence networks encode purely geometric information on a time series (similar to fractal dimensions and related concepts), which is distinctively different from most established methods of time series analysis that rely on temporal correlations between observations (including such that are commonly used for the quantitative analysis of recurrence matrices [11]). Consequently, recurrence networks capture different properties of attractors or, more generally, time series than othermethods of time series analysis. Recently, the detection of dynamical transitions in time series [12, 15] and invariant objects in phase space [13, 14] have turned out to be two very promising fields of application of these networks. In this work, we review the basic corresponding results and provide additional examples for both types of application.

2. Detecting invariant objects in phase space

Dynamically invariant objects in the phase space of complex systems, such as invariant manifolds or unstable periodic orbits, can be detected by considering the local vertex properties of recurrence networks. Basic examples for such measures are degree, closeness, and betweenness centrality. Specifically, the degree centrality k_v measures the number of direct neighbors of a vertex v with respect to a given spatial threshold distance ε, *i.e.,* it is proportional to the local phase space density. Closeness centrality c_v is related to the inverse mean network distance of a vertex with respect to all other vertices, implying that high values of closeness appear in the central parts of the attractor, whereas the outer parts are characterized by small values. Betweenness centrality b_v particularly highlights phase space regions with a low state density, which separate regions of higher density. Hence, it characterizes the local attractor fragmentation. Note that although betweenness and degree are not fully independent, they measure clearly distinct aspects of the phase space density [13, 14, 15].

For a fixed ε, all three centrality measures depend on the system size N. In contrast, the local recurrence rate $RR_v = k_v/(N-1)$ (*i.e.,* the density of connections in the vicinity of a vertex v) is a non-extensive property (*i.e.,* does not depend on N). Another non-extensive vertex property is the local clustering coefficient $\mathcal{C}_v$, which measures the presence of closed triangles in the network and, hence, characterizes localized higher-order spatial correlations between observations. Since recurrence networks are spatial networks, structures resolved by spatial variations of $\mathcal{C}_v$ correspond to a heterogeneous spatial filling of points. Specifically, high values of $\mathcal{C}_v$ often coincide with dynamically invariant objects, such as unstable periodic orbits or, more generally, invariant manifolds [13, 14, 15].

In order to illustrate the above general statements, we consider the behavior of the mentioned vertex properties for realizations of the logistic map at different values of the control parameter a (Fig. 1). One observes that the degree centrality is indeed directly proportional to the invariant density of the chaotic attractor, which explains the

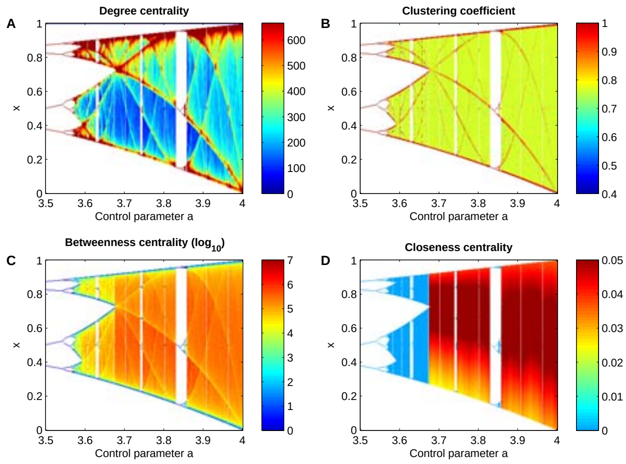

Figure 1: (A) Degree, (B) local clustering coefficient, (C) betweenness (in logarithmic units), and (D) closeness centrality for the ε-recurrence networks obtained from trajectories of the logistic map $x_{n+1} = ax_n(1-x_n)$ for different control parameters a ($N = 10,000$, no embedding, maximum norm, $\varepsilon = 0.05\sigma$ with σ being the empirical standard deviation of the considered realization).

sharp increase of k_v at the attractor boundaries and supertracks. b_v shows low values (*i.e.,* few shortest paths) close to the attractor boundaries, whereas the high values of k_v along the supertracks coincide with low betweenness values. The latter observation can be understood as an effect of the increasing redundancy of vertices for shortest paths in high-density regions of phase space. We further note that the supertracks are also resolved by $\mathcal{C}_v$, which is consistent with its interpretation as an indicator for dynamically invariant structures. Note that these considerations apply not only to maps, but also to continuous dynamical systems such as the Rössler, Lorenz [13, 14, 15], or Duffing systems (see Fig. 2)).

3. Detecting dynamical transitions in time series

One of the main applications of the quantitative analysis of recurrence matrices is the identification of dynamical transitions in time series. Specifically, if corresponding quantitative measures are calculated for individual, mutually overlapping parts of the time series [11], it is possible to use their variation with time for identifying changes in the dynamics of the underlying system. In a similar way, one may argue that sufficiently strong changes in the geometric properties of the associated attractor in phase space can be detected by measures obtained from recurrence networks. Recently, it has been demonstrated that a corresponding approach works indeed very well for detecting bifurcations in one-dimensional maps as well as real-world paleoclimate time series [12, 15]. Specifically, the average path length $\mathcal{L}$ has turned out to react sensitively to qualitative changes in the systems dynamics, whereas the global

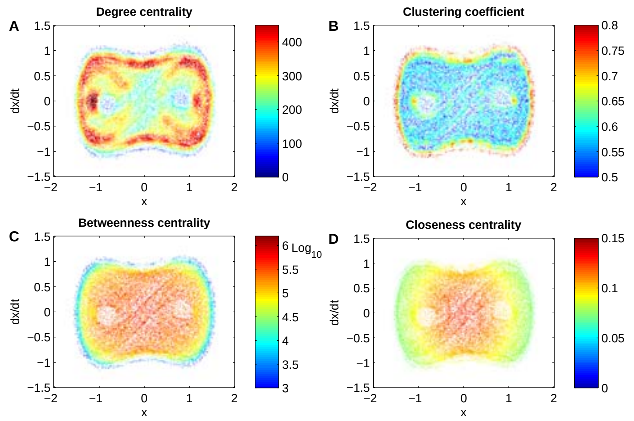

Figure 2: As in Fig. 1 for one trajectory of the Duffing system $\ddot{x} + \delta\dot{x} - \beta x + \alpha x^3 = \cos\omega t$ with $\alpha = \beta = 1.0$, $\delta = 0.2$, $\gamma = 0.36$ and $\omega = 1.0$ ($N = 30,000$, sampling time $\Delta t = 0.5$, $RR = 0.01$).

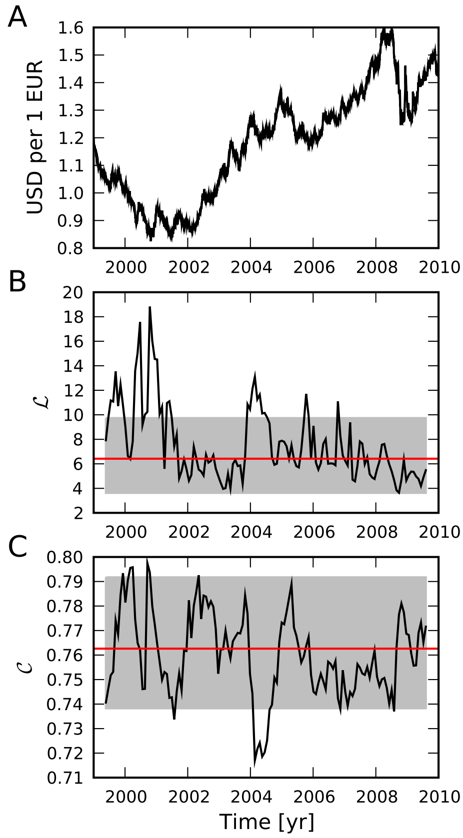

Figure 3: (A) Daily exchange rates between US Dollar and Euro, and (B,C) average path length and global clustering coefficient of the associated recurrence networks, obtained for sliding windows of 200 consecutive trading days (mutual offset of 20 days, no embedding, recurrence threshold ε chosen such that the total recurrence rate is preserved at 5% [13]). The shaded areas correspond to the lower and upper 5% confidence bounds obtained from the surrogate networks described in the text, the red line to the corresponding mean values.

clustering coefficient $\mathcal{C}$ is a good indicator for the presence of regular (*e.g.*, periodic) dynamics. In a similar way, it has been shown that network measures allow a better discrimination between periodic and chaotic dynamics in the parameter space of continuous-time dynamical systems than traditional recurrence quantification analysis [20].

In this work, we suggest another potential application for the quantitative analysis of recurrence networks. In economic time series, the behavior of the underlying system is typically influenced by both exogenous and endogenous shocks. Such shocks might be thought of not only causing abrupt changes in the mean and variance of the data set, but also in the qualitative appearence of its distribution. Considering the individual data as states in phase space, this implies that the geometric properties of the system are altered as well, which can be quantitatively characterized by measures computed from the associated recurrence networks.

Fig. 3 shows the temporal variations of global clustering coefficient and average path length of the recurrence networks obtained for the daily exchange rates between US Dollar and Euro over the last about 10 years (source: http://www.ecb.int/stats/exchange/eurofxref/html/index.en.html). In order to test the statistical significance, we additionally computed recurrence networks from surrogate data containing 10,000 random samples (containing the same number of data as the considered sliding windows) taken from the original data set [12, 15]. Our results reveal some interesting general features. For example, the pronounced minimum of the global clustering coefficient $\mathcal{C}$ in 2004 (panel (C)) appears to coincide with a period of increased volatility in the data (panel (A)). In a similar way, we find a significant minimum of $\mathcal{C}$ (panel (C)) in late 2001, possibly related with an increased market dynamics after the September 11 assaults. Multiple dynamical transitions, which appear to be related with change points in long-term trends, can be found in the years 2000-2001 and around early 2004, late 2005 and late 2006 when considering the variability of the average path length $\mathcal{L}$ (panel (B)). While the detection of this type of transition is fairly trivial, we point out that our method also allows identifying more subtle transitions, *e.g.*, by considering the exchange rates subjected to some appropriate detrending before further analysis. Note, however, that in case of the highly volatile multi-scale dynamics of financial time series, the features resolved after detrending crucially differ from those obtained from the original data. A similar behavior can be expected when additional embedding is used.

4. Summary

We have demonstrated that evolving complex networks constructed from the recurrence properties of dynamical systems allow studying non-trivial properties of these systems in terms of network-theoretic measures. Local vertex properties of recurrence networks (in particular, the local clustering coefficient) can be used for identifying dynamically invariant objects such as unstable periodic orbits or, more generally, invariant manifolds, whereas their global network properties (especially average path length and global clustering coefficient) detect time intervals corresponding to dynamical transitions and highly regular or volatile dynamics. Our findings open a wide field for novel applications of complex networks in data analysis, which make recurrence networks a promising candidate for the analysis of a variety of different transdisciplinary problems.

Acknowledgments

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