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On verified computation of laplacian eigenvalues over polygonal domain

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Abstract—For the eigenvalue problem of Laplace operator over general polygonal domain $\Omega \subset R^2$, we consider developing numerical method to give verified upper and lower bounds for the leading eigenvalues on polygonal domain with any shape. As all the error, such as the error in space approximation and the one in rounding computation of floating-point number, is totally estimated, the computation result is mathematically correct. Such kind of result can help us explore the solution of nonlinear partial differential equations.

1. Preliminaries

Let $\Omega \subset \mathcal{R}^2$ be bounded domain with polygon boundary, the eigenvalue problem of Laplace operator is to find u and $\lambda \in R$ such that

$$-\Delta u = \lambda u, \text{ on } \Omega \tag{1}$$

with u satisfying certain boundary conditions.

We perform the discussion in the framework of Sobolev spaces. Let V be subspace of $H^1(\Omega)$ with associated boundary condition, such as, homogeneous Dirichlet boundary condition or Neumann boundary condition. We do triangularization of Ω, denoted by $\mathcal{T}^h$, and use classical continuous piecewise linear finite element(FE) space $V_h(\subset V)$ as approximation space. The eigenvalue problem (1) can be characterized by variational form in certain V: Find $\lambda \in R$ and $u \in V$ such that ,

$$(\nabla u, \nabla v) = \lambda(u, v), \quad \forall v \in V \, . \tag{2}$$

The Ritz method is to solve the variational problem approximately in FE space V_h: Find $\lambda^h \in R$ and $u_h \in V_h$ such that

$$(\nabla u_h, \nabla v_h) = \lambda^h \ (u_h, v_h), \quad \forall v_h \in V_h \, . \tag{3}$$

Denote by $\{\lambda_i, u_i\}$ (resp. $\{\lambda_i^h, u_i^h\}$) the eigenpairs of (2) (resp. (3)) with eigenfunction to be althogonal with each other and the L_2 norm to be unit.

The objective of our research is to enclose $\{\lambda_k\}$'s by using approxiamte ones $\{\lambda_k^h\}$. The min-max principle will lead to an upper bound for λ_k, that is, $\lambda_k \le \lambda_k^h$, since $V_h \subset V$. But it is very difficult to find good computable lower bound for each λ_k, for which we develop new method to solve this problem.

Before futher talk, let's introduce one constant $C_{0,h}$ to be used later, which is related to function interpolations Π_K over triangle element K. For $u \in L_2(K)$, $\Pi_K u$ is constant function s.t.

$$\Pi_K u \equiv \int_K u(x)\, dx / \int_K 1 \, dx \, . \tag{4}$$

Global interpolations Π_h is just the extension of Π_K such that $\Pi_h u|_K = \Pi_K u$. Define h by the mesh size and $C_{0,h}$ the constants over triangulation $\mathcal{T}_h$,

$$C_{0,h} := \max_{K \in \mathcal{T}^h} C_0(K)/h \, , \tag{5}$$

where

$$C_0(K) := \sup_{v \in H^1(K) \backslash \{0\}} |\Pi_K u - u|_{L_2} / |u|_{H^1} \, .$$

2. Computable error estimation for FEM solution

Introduce projection operator P_h such that, for any $u \in V$, $P_h u \in V_h$ satisfies

$$(\nabla u - \nabla P_h u, \nabla v_h) = 0, \quad \forall v_h \in V_h \, . \tag{6}$$

Suppose $u \in V$ to be the solution of variational problem for $f \in L_2(\Omega)$,

$$(\nabla u, \nabla v) = (f, v), \quad \forall v \in V \, .$$

We will develop an a priori error estimate for $u - P_h u$ as below

$$\|u - P_h u\|_{L_2} \le M |u - P_h u|_{H^1} \le M^2 \|f\|_{L_2} \, .$$

Here M is a quantity dependent only on the triangulation of domain. Such a quantity is easy to evaluate in case domain is convex but very difficult for non-convex case. We solve this problem by adopting Raviart-Thomas FEM and piecewisely linear conforming FEM, which is based on the idea of F. Kikuchi [2].

Let W^h be the lowest order Raviart-Thomas FEM space over domain triangulation $\mathcal{T}^h$ and M^h the space of piecewise constant. Also, define subspace of W_h for f_h in M^h, $W^h_{f_h} := \{p_h \in W^h | \text{ div } p_h = f_h\}$. Notice that interpolation operator $\Pi_h : L_2(\Omega) \to M^h$ in Section 1 statisfying

$$(u - \Pi_h u, v_h) = 0, \quad \forall v_h \in M^h \, .$$

From the definition, we have $||u||_{L^2}^2 = ||\Pi_h u||_{L^2}^2 + ||u-\Pi_h u||_{L^2}^2$ and

$$||u - \Pi_h u||_{L^2} \le C_{0,h} h |u|_{H^1} \quad \text{if } u \in H^1(\Omega).$$

where $C_{0,h}$ is the constant defined in (5).

Let's introduce a computable quantity κ over finite dimensional spaces:

$$\kappa := \max_{f_h \in M^h \setminus \{0\}} \min_{v_h \in V_h} \min_{p_h \in W^h_{f_h}} ||p_h - \nabla v_h||_{L_2} / ||f_h||_{L_2} \quad (7)$$

The quantity κ is used to give error esimate for $u_i^* - P_h u_i^*$ as following.

Lemma 2.1 *Given $f_h \in M^h$, let $\tilde{u} \in H^1(\Omega)$ and $\tilde{u}_h \in V_h (\subset V)$ be the solutions of variational problems,*

$$(\nabla \tilde{u}, \nabla v) = (f_h, v), \quad \forall v \in V . \quad (8)$$
$$(\nabla \tilde{u}_h, \nabla v_h) = (f_h, v_h), \quad \forall v_h \in V_h , \quad (9)$$

respectively. Then we have a computable error estimate as below:

$$|\tilde{u} - \tilde{u}_h|_{H^1} \le \kappa ||f_h||_{L_2} . \quad (10)$$

Theorem 2.2 *For any $f \in L_2(\Omega)$, let $u \in V$ and $u_h \in V_h$ be solution of variational problems*

$$(\nabla u, \nabla v) = (f, v), \quad \forall v \in V, \quad (11)$$
$$(\nabla u_h, \nabla v_h) = (f, v_h), \quad \forall v_h \in V_h . \quad (12)$$

respectively. Introduce quantity $M := \sqrt{C_{0,h}^2 h^2 + \kappa^2}$, where $C_{0,h}$ is the constant defined in (5). Then, we have,

$$|u - u_h|_{H^1} \le M ||f||_{L_2}, \quad ||u - u_h||_{L_2} \le M^2 ||f||_{L_2} . \quad (13)$$

Lemma 2.3 *The quantity M, independent of f, will decrease when mesh is refined. By theoretical analysis, we can show that M tends to 0 in the same order as the error of linear conforming FEM solution solution.*

3. Lower and upper bound of eigenvalues

We then declare the theorem for a priori eigenvalue estimate, which is a combination of methods of F. Kikuchi and X. Liu [3] and Birkhoff, de Boor, Swartz and Wendroff [1]:

Theorem 3.1 *Let λ_k and λ_k^h the ones defined above. If $1 - M^2(\sum_{i=1}^k \lambda_i^2)^{1/2} > 0$, then*

$$\lambda_k^h \le \lambda_k + M^2 \sum_{i=1}^k \lambda_i^2 \Bigg/ \left(1 - M^2 (\sum_{i=1}^k \lambda_i^2)^{1/2}\right)^2 . \quad (14)$$

Denote by $\phi_k(\lambda_k; \lambda_1, \cdots, \lambda_{k-1})$ the right side of equation (14), which is monotonically increasing on λ_k when the parameters $\{\lambda_1, \cdots, \lambda_{k-1}\}$ are fixed. Thus we can have a posteriori error estimate for λ_k as below

$$\lambda_k \ge \phi_k^{-1}(\lambda_k^h; \lambda_1, \cdots, \lambda_{k-1}).$$

Therefore, we can bound egienvalues $\{\lambda_k\}$'s recursively. Since $\lambda_{k,h}$ works as upper bound of λ_k, we can replace λ_k in the second term of (14) by $\lambda_{k,h}$, that is,

$$\lambda_k^h - M^2 \sum_{i=1}^k \lambda_{i,h}^2 \Bigg/ \left(1 - M^2 (\sum_{i=1}^k \lambda_{i,h}^2)^{1/2}\right)^2 \le \lambda_k . \quad (15)$$

The result of Birkhoff and further the one in Theorem 3.1 are based on Min-Max principle. Recently, by applying Max-Min pripnle directly, we obtained an improved result.

Theorem 3.2 *Let λ_k and λ_k^h the ones defined above. For any $k \ge 1$, we have*

$$\lambda_k^h / (1 + M^2 \lambda_k^h) \le \lambda_k . \quad (16)$$

4. Computation results

In a summary, we evaluate the eigenvalues of laplacian in a framewok as below.

1) Triangulate the domain Ω and create finite element space V_h.
2) Solve matrix eigenvalue problem $A^h x = \lambda^h B^h x$ corresponding to V_h.
3) Evaluate quantity M related to the mesh and domain.
4) Calculation of the lower / upper bounds for λ_k by using (16).

In the following, we will talk about the efficiency and flexibility of our proposed algorithm by solving several eigenvalue problems with different boundary condition.

4.1. Triangle domain

We consider the eigenvalue problem over unit isosceles right triangle. By $\{\lambda_{D,i}, u_{D,i}\}$ and $\{\lambda_{N,i}, u_{N,i}\}$, we denote the laplacian's eigenpairs in cases of homogeneous Dirichlet boundary and homogeneous Neumann boundary, respectively. Due to the symmetry of specified triangle domain, we can apply reflecting techniques, see for instance, [6], to obtain the explicit eigenpairs as below,

$$\{\lambda_D = m^2 + n^2, u_D = \sin m\pi x \sin n\pi y - \sin n\pi x \sin m\pi y\}_{m > n \ge 1},$$

$$\{\lambda_N = m^2 + n^2, u_N = \cos m\pi x \cos n\pi y + \cos n\pi x \cos m\pi y\}_{m \ge n \ge 0}.$$

$\lambda_{N,i}$	exact value	$h = 1/64$	
		lower	upper
1	$5\pi^2 \approx 49.348$	49.254	49.400
2	$10\pi^2 \approx 98.696$	98.352	98.930
3	$13\pi^2 \approx 128.305$	127.687	128.662
4	$17\pi^2 \approx 167.783$	166.746	168.413
5	$20\pi^2 \approx 197.392$	196.129	198.439

Table 1: Eigenvalue estimates for laplacian on triangle domain (Dirichlet b.d.c.)

$\lambda_{N,i}$	exact value	$h = 1/64$	
		lower	upper
2	$\pi^2 \approx 9.8696$	9.8658	9.8716
3	$2\pi^2 \approx 19.739$	19.728	19.752
4	$4\pi^2 \approx 39.478$	39.418	39.511
5	$5\pi^2 \approx 49.348$	49.283	49.428

Table 2: Eigenvalue estimates for laplacian on triangle domain (Neumann b.d.c.)

4.2. L-shaped domain

Let domain be $\Omega = [0,2] \times [0,2] \setminus [1,2] \times [1,2]$. The boundary condition is selected to be homogeneous Dirichlet condition. In Table 4, we list the first 5 approximate eigenvalues obtained with refined mesh and the verified lower and upper bounds by our proposed method. Letting the lower and upper bounds of eigenvalue to be $\lambda_{low}, \lambda_{up}$, the *relative error* is computed as $2(\lambda_{up} - \lambda_{low})/(\lambda_{up} + \lambda_{low})$.

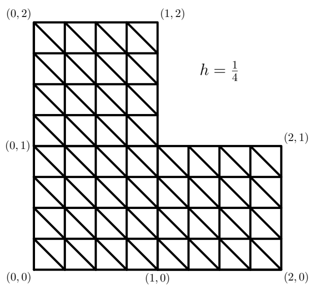

Figure 1: Uniform mesh of L-shaped domain

h	κ	$C_{0,h}$	M	order of κ
1/4	0.1466	0.080	0.1668	-
1/8	0.0882	0.040	0.0968	0.73
1/16	0.0538	0.020	0.0574	0.71
1/32	0.0332	0.010	0.0348	0.70

Table 3: Values of κ in case of L-shaped domain

4.3. Domain with crack

Let Ω be a unit square domain with crack $\{(x, 0.5) | 0 < x < 0.5\}$. Divide the boundary by two parts as below:

$$\partial_D\Omega = \partial\Omega \cap \{(x,y) | y = 1 \text{ or } y = 0 \text{ or } x = 1\}, \partial_N\Omega = \partial\Omega \backslash \partial_D\Omega.$$

We solve the eigenvalue problem associated with mixed boundary condition :

$$-\Delta u = \lambda u \text{ in } \Omega, \quad u = 0 \text{ on } \partial_D\Omega, \quad \partial u/\partial n = 0 \text{ on } \partial_N\Omega\,.$$

In the computation, we adopted non-uniform triangulation, which has denser mesh around center point (0.5, 0.5). The

λ_i	lower	approx.	upper	relative error
1	9.5585	9.6397	9.6698	0.012
2	14.950	15.361	15.225	0.018
3	19.326	19.739	19.787	0.024
4	28.605	29.521	29.626	0.035
5	30.866	31.913	32.058	0.038

Table 4: Eigenvalue evaluation for L-shaped domain ($h = 1/32$)

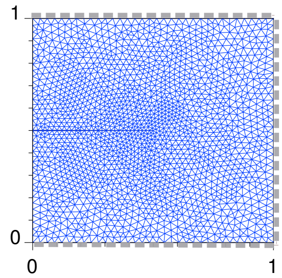

Figure 2: Domain of crack with mixed-boundary condition

result is displayed in Table 5. By theoretical analysis, it is easy to shown that the following values are the part of the exact eigenvalues.

$$\{\frac{5}{4}\pi, \frac{13}{4}\pi^2, \frac{29}{4}\pi^2\} \approx \{12.337, 32.076, 71.555\}$$

We can observe that numerical computation gives satisfying estimates. The eigenfunctions for the leading four eigenvalues are displayed in Figure 3.

λ_i	lower bound	upper bound	relative error
1	12.233	12.343	0.009
2	16.087	16.276	0.012
3	31.392	32.119	0.022
4	51.049	52.998	0.037
5	68.241	71.768	0.050

Table 5: Eigenvalue estimation for domain of crack ($M = 0.027$)

5. Conclusion

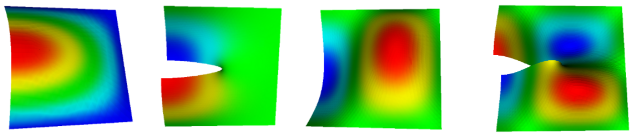

Figure 3: Eigenfunctions corresponding to leading four eigenvalues

Our proposed method makes it possible to evaluate eigenvalues of Laplacian over polygonal domain of any shape. This result can help us give norm estimate for differential opertaors by evaluating the eigenvalue with smallest absolute value, which is important part in in verifying the solution existence for non-linear elliptic differential equations. Such kind of application can be found in this proceding with title as *Computer assisted proofs for solutions to nonlinear elliptic partial differential equations on arbitrary polygonal domain* by Akitoshi Takayasu, Xuefeng Liu and Shin'ichi Oishi.

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